The initial value problem for semilinear Schrödinger equations

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Chapter 1

Introduction

The present thesis is concerned with the initial value problem for semilinear Schrödinger equations of the form:

$$\partial_t u - i\Delta u = F(u, \nabla u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where $u(t,x)$ is $\mathbb{C}$-valued, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1,\cdots,N)$, $\nabla = (\partial_1,\cdots,\partial_N)$, $\Delta = \partial_1^2 + \cdots + \partial_N^2$, $i = \sqrt{-1}$ and $N \in \mathbb{N}$ is the spatial dimension. Throughout the present thesis, we deal with the power nonlinearlity. More precisely, we assume that the nonlinear term $F(u,q) : \mathbb{C}\times\mathbb{C}^N \to \mathbb{C}$ satisfies

$$\begin{aligned} F(u,q) &\in C^\infty(\mathbb{R}^2\times\mathbb{R}^{2N};\mathbb{C}),\\ F(u,q) &= O\left(|u|^\rho + |q|^\rho\right) \quad \text{near} \quad (u,q) = 0 \end{aligned}$$

with some integer $\rho \geq 2$. The variable q corresponds to $\nabla u = (\partial_1 u,\cdots,\partial_N u)$. In the same way as the complex analysis, we define $\partial/\partial u$, $\partial/\partial\bar{u}$, $\partial/\partial q_j$ and $\partial/\partial\bar{q}_j$ by

$$\frac{\partial}{\partial u} = \frac{1}{2}\Big(\frac{\partial}{\partial v_0} - i\frac{\partial}{\partial w_0}\Big), \qquad \frac{\partial}{\partial \bar{u}} = \frac{1}{2}\Big(\frac{\partial}{\partial v_0} + i\frac{\partial}{\partial w_0}\Big),$$

$$\frac{\partial}{\partial q_j} = \frac{1}{2}\Big(\frac{\partial}{\partial v_j} - i\frac{\partial}{\partial w_j}\Big), \qquad \frac{\partial}{\partial \bar{q}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial v_j} + i\frac{\partial}{\partial w_j}\Big),$$

$$v_0 = \mathrm{Re}\, u,\ w_0 = \mathrm{Im}\, u,\ v_j = \mathrm{Re}\, q_j,\ w_j = \mathrm{Im}\, q_j,\ j = 1,\cdots,N.$$

Now we will give a few remarks on the technical terms. In the theory of partial differential equations, we usually call the nonlinearity of the equation (1.2) and the problem (1.1)–(1.2) semilinear and the initial value problem respectively. Someone calls (1.1) nonlinear Schrödinger equation and another one calls (1.1)–(1.2) the Cauchy problem. In mathematical physics, the word "nonlinear Schrödinger equation" means semilinear

Schrödinger equation whose nonlinear term $F(u,q)$ is independent of q. Especially, in the soliton theory, the nonlinear term of nonlinear Schrödinger equation is $F(u,q) = i\gamma|u|^2u$, where $\gamma \in \mathbb{R}\setminus\{0\}$. On the other hand, the word "Cauchy problem" means sometimes the initial value problem for kowalewskian-type of partial differential equation:

$$\partial_t^m u - \sum_{\substack{|\alpha|+j\le m \\ j<m}} a_{\alpha,j}(t,x)\partial^\alpha\partial_t^j u = f(t,x) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N,$$

$$\partial_t^j u(0,x) = u_j(x) \quad (j = 0,1,\cdots,m-1) \qquad \text{in} \quad \mathbb{R}^N.$$

Here m is a positive integer. Kowalewskian-type of partial differential operators requires that $|\alpha| + j$ is less than or equal to the heiest order of the differentiation with respect to the time variable t.

The past two decades, studies on the completely integrable systems have been very popular in mathematical physics. Someone presented or obtained new integrable systems and another one proved that certain systems were completely integrable formally or mathematically. Most of partial differential equations and systems appearing in this subject model the phenomena of the propagation of disspersive waves. Roughly speaking, the dispersive wave means that its propagation speed changes in accordance with its frequency.

As typical examples of such equations and systems, we cite the Korteweg-de Vries equation

$$\partial_t v + \partial^3 v + v\partial v = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R},$$

the Benjamin–Ono equation

$$\partial_t v + |D|\partial v + v\partial v = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R},$$

the Boussinesq equation

$$\partial_t^2 v + \partial^4 v + \partial^2 v \pm \partial v\partial^2 v = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R},$$

the nonlinear Schrödinger equation

$$\partial_t w - i\partial^2 w \pm i|w|^2 w = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \tag{1.3}$$

and the elliptic–elliptic Davey–Stewartson equation

$$\partial_t w - i\Delta w = i|w|^2 w + iR_1^2\left(|w|^2\right)w \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^2,$$

where v is $\mathbb{R}$-valued, w is $\mathbb{C}$-valued, $|D| = (-\partial^2)^{1/2}$ and $R_1 = \partial_1/|D|$ is the Riesz transformation. The above equations can be seen as

$$\left(I_n\partial_t + ip(D)\right)U = F(U,\partial U,\cdots,\partial^{m-1}U) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^N, \tag{1.4}$$

where $U(t,x)$ is $\mathbb{R}^n$ or $\mathbb{C}^n$-valued ($n = 1,2,3,\cdots$), I_n is the $n\times n$ identity matix, $p(\xi)$ is a $n\times n$ matrix-valued symbol satisfying $p(\xi) \in (S^m)^{n\times n}$ and $p(\xi) - {}^t\overline{p(\xi)} \in (S^{m-1})^{n\times n}$, and $F(U,\partial U,\cdots,\partial^{m-1}U)$ $(m > 1)$ is a nonlinear term. For the definition of the class S^m, see Section 2.3

To illustrate the dispersive property, we consider the simplest Schrödinger equation in one space dimension

$$\partial_t w - i\partial^2 w = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \tag{1.5}$$

and we will find the plane-wave solution to (1.5). The substitution of

$$w(t,x) = e^{i(ct+\xi x)} \quad (c\in\mathbb{C},\xi\in\mathbb{R})$$

into (1.5) imples the relationship between the propagation speed c and the frequency ξ:

$$c(\xi) = -\xi^2. \tag{1.6}$$

This asserts that c must be a real number and that the larger the frequency ξ is, the larger the propagation speed c becomes. In mathematical physics a relationship like (1.6) is called the dispersion-relation. Then, let us call equations and systems like (1.4) nonlinear dispersive equations and systems respectively.

Some of nonlinear dispersive equations and systems cannot allow the classical energy estimates. In other words, the regularity of solutions to these equations and systems cannot follow from the usual L^2-type estimates. This means that it is difficult to solve the initial value problems for these equations and systems in C^∞-category. We remark here that we can always get analytic solutions to such problems by means of so-called abstract Cauchy–Kowalewski theorem. More precisely, if the nonlinear term is real-anaytic with respect to all variables, and if the initial data is analytic in some product domain of strips containing real lines, then we can get a unique solution which is analytic with respect to the spatial variables in the product domain of small strips.

To explain this situation, let us consider the initial value problem for linear Schrödinger equations with constant coefficient:

$$\partial_t w - i\partial^2 w + i\alpha\partial w = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \tag{1.7}$$

$$w_0(x) = 0 \quad \text{in} \quad \mathbb{R}, \tag{1.8}$$

where α is a positive constant. Let $\phi(\xi)$ satisfy

$$\phi(\xi) = \begin{cases} 1, & \xi\in(0,1), \\ 0, & \text{otherwise}. \end{cases}$$

We put $\phi_n(\xi) = \phi(\xi - n)$, $n = 1,2,3,\cdots$. Formally, the solution to (1.7)–(1.8) is written as

$$w(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ix\xi - it\xi^2 + \alpha\xi t}\hat{w}_0(\xi)d\xi, \tag{1.9}$$

$$\hat{w}_0(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ix\xi}w_0(x)dx.$$

If we put $w_0(\xi) = \phi_n(\xi)$, then the Plancherel–Parseval formula implies

$$
\begin{aligned}
\left\|w(t)\right\|_{L^2}^2 &= \left\|w(t)\right\|_{L^2}^2 \\
&= \int_{-\infty}^{+\infty} e^{2\alpha\xi t}\left|\phi_n(\xi)\right|^2 d\xi \\
&= \int_n^{n+1} e^{2\alpha\xi t} d\xi \\
&\geq e^{2\alpha n t} \quad \text{for} \quad t \geq 0. \qquad (1.10)
\end{aligned}
$$

(1.10) asserts that for any large constant $R > 0$ and for any short time $t > 0$, there exists an entire function $w_0(x)$ such that $\|w_0\|_{L^2} = 1$ and $\left\|w(t)\right\|_{L^2} \geq R$. When either α or t is a negative number, the same is true provided that $n \in \mathbb{Z}$ is negative and $|n|$ is sufficiently large. This example shows that if the imaginary part of the coefficient of the first order term α does not vanish, then the initial value problem (1.7)-(1.8) is not L^2-well-posed. Namely, we can solve (1.7)-(1.8) only in some analytic function space unless $\alpha = 0$. This fact is very closed to the method of the abstract Cauchy–Kowalewski theorem.

In mathematical physics, threre are some nonlinear dispersive equations which do not allow the classical energy estimates. As examples, we cite the (generalized) Heisenberg ferromagnetic model

$$\partial_t u - i\Delta u = \frac{2iu}{1+|u|^2}\sum_{j=1}^{N}(\partial_j u)^2 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^N, \qquad (1.11)$$

the Zakharov equation

$$\partial_t u - i\Delta u = \left(\int_0^t \frac{\sin\{|D|(t-\tau)\}}{|D|}\Delta|u(\tau)|^2 d\tau\right) u + |u|^2 u \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^N, \qquad (1.12)$$

the long wave–short wave resonance equation

$$\partial_t u - i\partial^2 u = \left(\int_0^t \partial|u(\tau)|^2 d\tau\right) u + |u|^2 u \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \qquad (1.13)$$

and the elliptic–hyperbolic Davey–Stewartson equation

$$
\begin{aligned}
\partial_t u - i(\partial_1^2 + \partial_2^2)u &= i\alpha_0|u|^2 \\
&+ i\alpha_1\left(\int_{x_2}^{+\infty}\partial_1|u(t,x_1,x_2')|^2 dx_2'\right) u \\
&+ i\alpha_2\left(\int_{x_1}^{+\infty}\partial_1|u(t,x_1',x_2)|^2 dx_1'\right) u \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^2, \qquad (1.14)
\end{aligned}
$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ are constants. We remark here that the original Heisenberg ferromagnetic model is formulated by

$$\partial_t S = S \wedge \Delta S \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^N, \qquad (1.15)$$

where $S(t,x)$ is S^2-valued and $\wedge$ denotes the wedge product in $\mathbb{R}^3$. When $S(t,x)$ is close to some constant $S_0 \in S^2$ uniformly in $\mathbb{R}\times\mathbb{R}^N$, (1.15) becomes (1.11) by the stereographic representation.

The nonlinear terms of these equations contain ∇u. Moreover, some of them are nonlocal as nonlinear operators. Then, it is not so easy to prove even local existence theorems for these equations. Local and global existence, stability or instability of special solutions in which some physists are interested, regularity of solutions (so-called local smoothing property), and etc are important and interesting not only in mathematical analysis but also in mathematical physics. A lot of authors studied global existence theorems formally or rigourously from a view point of the inverse scattering thechnique (see, e.g., [13]). But there are few works concerned with these problems via the theory of partial differential equations.

The purpose of the present thesis is to prove local and global existence theorems for the initial value problem (1.1)–(1.2). We see it as a typical example of the initial value problems for nonlinear dispersive equations and systems, which cannot allow the classical energy estimates of solutions. Because our nonlinear term $F(u, \nabla u)$ satisfies the local property, (1.1)–(1.2) is somewhat easier than nonlocal nonlinear terms like (1.12) and (1.14). Moreover, we study the initial value problem for (1.14).

The present thesis is a rearrangement of the author's papers [2], [3], [4], [5], [6], [7] and contains a few improvemnt of them. The organization is as follows. In Chapter 2 we introduce the outline of the theory of linear Schrödinger-type equations, and modify this theory so that it can be applied to solve semilinear equations. Chapter 2 is based on [2], [3] and [6]. Chapter 3 is devoted to preliminaries: estimates on nonlinear term and parabolic regularlization, which are contained in [2], [3], [4] [5] and [6]. In Chapter 4 we present local existence theorems which were proved in [2], [3] and [6]. Our proofs consists of the parabolic regularization and the uniform estimates of solutions. The later follows from some linear estimates which are obtained in Chapter 2. In Chapter 5 we study global existence theorems which were proved in [4], [5] and [6]. Our strategy of the proofs is based on the local existence theorems in Chapter 4 and the *a priori* estimates, where the decay estimates in Chapter 3 play an essential role. Finally, in Chapter 6 we present local and global existence theorems of the initial value problem for the elliptic–hyperbolic Davey–Stewartson equation. For this purpose, we make straightforward use of the smoothig property of $e^{it\Delta}$, because our linear estimate developed in Chapter 2 does not work well. The material of the final chapter is [7].

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Notations

Here we summarize notations used thoughout of this thesis except for Chapter 6, where we make use of another notations.

$\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. $D_j = -i\partial_j$, $D = (D_1, \cdots, D_N) = -i\nabla$, $\partial_{\xi_j} = \partial/\partial\xi_j$, $\partial_\xi = (\partial_{\xi_1}, \cdots, \partial_{\xi_N})$, for $j = 1, \cdots, N$.

$$\begin{aligned} J_k u &= e^{ix_k^2/4(1+t)} 2i(1+t)\partial_k\Big(e^{-ix_k^2/4(1+t)}u\Big) = (x_k + i(1+t)\partial_k)u, \\ J_k' u &= e^{ix_k^2/4t} 2it\partial_k\Big(e^{-ix_k^2/4t}u\Big) = (x_k + i(1+t)\partial_k)u, \end{aligned}$$

for $k = 1, \cdots, N$, where $\theta(t, x_k) = x_k^2/4(1+t)$. $J = (J_1, \cdots, J_N)$. $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$, $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$, $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N}$, $J^\alpha = J_1^{\alpha_1} \cdots J_N^{\alpha_N}$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, for any multi index $\alpha = (\alpha_1, \cdots, \alpha_N) \in (\mathbb{Z}_+)^N$. $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$, $\langle\xi_j\rangle = \sqrt{1 + \xi_j^2}$, $\langle D\rangle = (1 - \Delta)^{1/2}$, $\langle D_j\rangle = (1 - \partial_j^2)^{1/2}$.

$$W^{s,r} = W^{s,r}(\mathbb{R}^N) = \left\{u \in \mathcal{S}'(\mathbb{R}^N) \;\Big|\; \|u\|_{W^{s,r}} = \left(\int_{\mathbb{R}^N} |\langle D\rangle^s u|^r dx\right)^{1/r} < +\infty\right\},$$

$L^r = W^{0,r}$, $H^s = W^{s,2}$ for $s \in \mathbb{R}$ and $1 \le r < \infty$.

$$W^{s,\infty} = W^{s,\infty}(\mathbb{R}^N) = \left\{u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{W^{s,\infty}} = \text{ess.sup}|\langle D\rangle^s u| < +\infty\right\},$$

$L^\infty = W^{0,\infty}$, for $s \in \mathbb{R}$.

$$H^{s,s'} = H^{s,s'}(\mathbb{R}^N) = \left\{u \in \mathcal{S}'(\mathbb{R}^N) \mid \|\langle x\rangle^{s'}\langle D\rangle^s u\|_{L^2} < +\infty\right\}$$

for $s, s' \in \mathbb{R}$. $\|\cdot\|_s$ means $(H^s)^2$-norm. In particular, $\|\cdot\|$ and $(\cdot,\cdot)$ mean $(L^2)^2$-norm and $(L^2)^2$-inner product respectively. $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^N)$ is the set of all distributions on $\mathbb{R}^N$. $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^N)$ denote the Schwartz class and its topological dual respectively. The operator $\langle D\rangle^s$ is defined by

$$\langle D\rangle^s u = (2\pi)^{-N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} e^{i(x-y)\cdot\xi}\langle\xi\rangle^s u(y)dyd\xi \quad \text{for} \quad u \in \mathcal{S},$$

and is extended on $\mathcal{S}'$ via the duality. $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}^N)$ is the set of all C^∞-functions on $\mathbb{R}^N$, whose derivatives of any order are all bounded. When X and Y are normed vector spaces, $\mathcal{L}(X, Y)$ and $\mathcal{L}(X)$ are the set of all bounded linear operators from X to Y and from X to X respectively. $\hat{x}_j = (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_N)$, for $x \in \mathbb{R}^N$ and $j = 1, \cdots, N$. δ_{jk} is Kronecker's delta, i.e., $\delta_{jk} = 1$ if $j = k$, $\delta_{jk} = 0$ otherwise. $[s]$ means the largest integer less than or equal to $s \ge 0$. Different positive constants might be denoted by the same letter C.

Chapter 2

Linear estimates

2.1 Theory of linear Schrödinger-type equations

In this section we introduce the theory of linear Schrödinger-type equations. This theory plays an essential role to obtain the energy estimates of solutions to semilinear equations. We start with the inital value problem of the form:

$$Su \equiv \Big(\partial_t - i\Delta + \sum_{j=1}^{N} b_j(x)\partial_j + c(x)\Big)u = f(t,x) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^N, \tag{2.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{2.2}$$

where the coefficients $b_1(x), \cdots, b_N(x)$ belong to $\mathcal{B}^\infty(\mathbb{R}^N)$.

The purpose of the linear theory is to characterize Schrödinger-type operators. To explain the detail, we introduce the notion of well-posedness. Let X be a Fréchet space contained in $\mathcal{D}'(\mathbb{R}^N)$. We say that the initial value problem (2.1)–(2.2) is X-well-posed in the both direction of $t \in \mathbb{R}$ if and only if for any $u_0(x) \in X$ and for any $f(t,x) \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$, there exists a unique solution $u \in C(\mathbb{R}; X)$ to (2.1)–(2.2). In view of Banach's closed graph theorem, this is equivalent to the apparently stronger condition, that is for any $u_0(x) \in X$ and for any $f(t,x) \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$, there exists a unique solution $u \in C(\mathbb{R}; X)$ to (2.1)–(2.2) and the map $(u_0, f) \mapsto u$ is continuous linear operator from $X \times L^1_{\mathrm{loc}}(\mathbb{R}; X)$ to $C(\mathbb{R}; X)$. According to S. Mizohata ([35]), S is a Schrödinger-type operator if and only if (2.1)–(2.2) is H^∞-well-posed in the both direction of $t \in \mathbb{R}$.

Historically, first J. Takeuchi ([47], [48]) studied the necessary condition of L^2-well-posedness. After some improvement of S. Mizohata ([36], see also [35]), the necessary condition is

$$\sup_{\substack{x\in\mathbb{R}^N \\ \omega\in S^{N-1} \\ t\in\mathbb{R}}} \left| \mathrm{Im} \int_0^t \sum_{j=1}^{N} b_j(x+\omega\tau)\omega_j \, d\tau \right| < +\infty. \tag{2.3}$$

We note here that when $N = 1$, (2.3) becomes

$$\sup_{x\in\mathbb{R}^N} \left|\operatorname{Im} \int_0^x b(y)dy\right| < +\infty, \tag{2.4}$$

where $b(x) = b_1(x)$. Roughly speaking, (2.3) means that $\operatorname{Im} b_1(x), \cdots, \operatorname{Im} b_N(x)$ must be improperly Riemannian integrable for any line in $\mathbb{R}^N$. In particular, (2.4) is the necessary and sufficient condition for L^2-well-posedness. In fact, if we assume $N = 1$ and (2.4), then a gauge transformation

$$u(x) \longmapsto v(x) \equiv u(x)k_0(x), \tag{2.5}$$

$$k_0(x) = \exp\left(\frac{i}{2}\int_0^x b(y)dy\right)$$

is an automorphism in $L^2(\mathbb{R})$, and (2.1)–(2.2) becomes

$$\left(\partial_t - i\partial^2 + \tilde{c}(x)\right)v = k_0(x)f(t,x) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \tag{2.6}$$

$$v(0,x) = u_0(x)k_0(x) \quad \text{in} \quad \mathbb{R}, \tag{2.7}$$

Since the first order term $b(x)\partial$ is canceled out by the commutator $[k_0(x), -i\partial^2]/k_0(x)$, one can easily solve (2.6)–(2.7). Thus, this shows that (2.4) is also the sufficient condition of L^2-well-posedness.

Secondly, S. Mizohata ([36], see also [35]) studied the sufficient condition of L^2-well-posedness with general space dimensions. He developed the above idea directly. Under the appropriate condition which is stronger than the necessary condition (2.3), he constructed a transformation $u \mapsto K(t)u$ such that this was an automorphic in $L^2(\mathbb{R}^N)$ and the commutator $[K(t), -i\Delta]K(t)^{-1}$ coinsided with $\sum b_j(x)\partial_j$. In this case, $K(t)$ is a pseudo-differential operator which belongs to $OPS^0_{0,0}(\mathbb{R}^N\times\mathbb{R}^N)$ (see [33]).

Thirdly, developping the ideas of J. Takeuchi and S. Mizohata, W. Ichinose ([24] and [25]) studied H^∞-well-posedness. His results are natural extension of L^2-well-posedness in a sense.

We remark here that if $\operatorname{Im} b_1(x), \cdots, \operatorname{Im} b_N(x)$ satisfy

$$\partial_j\left(\operatorname{Im} b_k(x)\right) - \partial_k\left(\operatorname{Im} b_j(x)\right) = 0 \quad \text{for} \quad x\in\mathbb{R}^N, j,k = 1,\cdots,N, \tag{2.8}$$

then a gauge transformation

$$u(x) \longmapsto v(x) \equiv u(x)\exp\left(-\frac{1}{2}\int_0^1 \sum_{j=1}^N \operatorname{Im} b_j(sx)x_j ds\right) \tag{2.9}$$

is also available in the same way as (2.6)–(2.7). When $N = 1$, (2.8) is satisfied automatically. S. Tarama ([46]) obtained the necessary and sufficient condition for H^∞-well-posedness under the condition (2.8) by using the gauge transformation (2.9). In the same way, we can prove that (2.3) is the necessary and sufficient condition for L^2-well-posedness under the condition (2.8). Some of semilinear Schrödinger equations satisfy (2.8). In terms of semilinear equations, (2.8) becomes

$$\partial_j\left(\operatorname{Im}\frac{\partial F}{\partial q_k}(u,\nabla u)\right) - \partial_k\left(\operatorname{Im}\frac{\partial F}{\partial q_j}(u,\nabla u)\right) = 0 \quad \text{for} \quad u\in C^1(\mathbb{R}^N), j,k = 1,\cdots,N, \tag{2.10}$$

which is equivalent to the condition: there exists a $\mathbb{R}$-valued function $G(u)$ such that

$$\partial_j G(u) = \operatorname{Im}\frac{\partial F}{\partial q_j}(u,\nabla u) \quad \text{for} \quad u\in C^1(\mathbb{R}^N), j = 1,\cdots,N. \tag{2.11}$$

In this case, (2.9) becomes

$$u(x) \longmapsto v(x) \equiv u(x)\exp\left(-\frac{1}{2}G\big(u(x)\big)\right). \tag{2.12}$$

Using (2.12), A. Soyeur ([44]) and T. Ozawa ([39]) studied local and global existence theorems for such semilinear Schrödinger equations.

As we have seen before, the basic idea for linear Schrödinger-type equations is to find an automorphism $u \mapsto K(t)u$ in L^2 or H^∞ such that the commutator $[K(t), -i\Delta]K(t)^{-1}$ eliminates the first order term $\sum \operatorname{Im} b_j(x)\partial_j$ in a sense. We remark here that $[K(t), -i\Delta]K(t)^{-1}$ does not have to coinside with $\sum \operatorname{Im} b_j(x)\partial_j$ or $\sum b_j(x)\partial_j$ necessarily.

Recently, S. Doi ([11], [12]) developed new method to solve linear Schrödinger-type equations. His method requires only the integrability of $\operatorname{Im} b_1(x), \cdots, \operatorname{Im} b_N(x)$ in any line in x, which is slightly stronger the condition (2.3). His idea is to find an automorphism $u \mapsto K(t)u$ in $L^2(\mathbb{R}^N)$ such that $[K(t), -i\Delta]K(t)^{-1}$ is a first order elliptic psuedo-differential operator which is stronger than the first order partial differential operator $\sum \operatorname{Im} b_j(x)\partial_j$. We remark here that his transformation operator belongs to the class $OPS^0_{1,0}$. In general, $(1,0)$-type operators are easier to handle than $(0,0)$-type ones. As we will show later, his idea is very useful to solve semilinear equations. There are two advantage of his method. One is the simplicity of the condition on $\operatorname{Im} b_1(x), \cdots, \operatorname{Im} b_N(x)$, and another is the type of transformation operator.

When we apply the linear theory to solving semilinear equations, we treat (1.1) as a system for ${}^t(u,\bar{u})$ because the nonlinear term $F(u,\nabla u)$ contains not only ∇u but also $\nabla\bar{u}$. Then, in Sections 2.2 and 2.3 we study the initial value problem for some 2×2 linear systems, which correspond to the couple of equations (1.1) and its complex conjugate. Section 2.2 is devoted to one dimensional case, which can be treated by the gauge transformation. In Section 2.3 we study the case of general space dimensions. There, the diadonalization technique and S. Doi's method play essential role. The former is well-known in the theory of regularly hyperbolic systems.

2.2 Linear estimates in one space dimension

In this section we study the initial value problem for 2×2 linear systems of the form:

$$L_1 u \equiv \Big(I\partial_t + iH_1(t)\Big)v = f(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}, \tag{2.13}$$

$$v(0,x) = v_0(x) \quad \text{in} \quad \mathbb{R}, \tag{2.14}$$

where $v = {}^t(v_1, v_2)$ is $\mathbb{C}^2$-valued, $f = {}^t(f_1, f_2)$, I is the 2×2 identity matrix and $H_1(t)$ is defined as

$$H_1(t) = \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} - i\begin{bmatrix} b_1(t,x) & b_2(t,x) \\ \overline{b_2(t,x)} & \overline{b_1(t,x)} \end{bmatrix}\partial$$

$$b_1(t,x), b_2(t,x) \in C^1\Big([0,T]; \mathcal{B}^\infty(\mathbb{R})\Big).$$

We remark here that it is sufficient to study the forward initial value problem.

The purpose of the present section is to mention the well-posedness of the initial value problem (2.13)–(2.14). More precisely, we present a certain sufficient condition on L^2-well-posedness. We do not investigate the necessary and sufficient condition on L^2-well-posedness. However, it is enough to apply to solving semilinear equations. In general, the necessary and sufficient condition for well-posedness in the linear theory is not enough to apply to nonlinear problems, because the nonlinearity gives very complex structure as a partial differential operator. We keep a semilinear equation and its complex conjugate

$$\begin{aligned} \partial_t u - i\partial^2 u + b_1(t,x)\partial u + b_2(t,x)\partial u &= f_1(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}, \\ \partial_t u + i\partial^2 \bar{u} + \overline{b_1(t,x)}\partial\bar{u} + \overline{b_2(t,x)}\partial\bar{u} &= \overline{f_1(t,x)} \quad \text{in} \quad (0,T)\times\mathbb{R}, \end{aligned}$$

in mind. In view of (2.4), we assume

$$M_1 \equiv \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} \left|\int_0^x \operatorname{Im} b_1(t,y)dy\right| < +\infty, \tag{2.15}$$

$$M_2 \equiv \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} \left|\int_0^x \operatorname{Im} \partial_t b_1(t,y)dy\right| < +\infty. \tag{2.16}$$

Because the coefficients $b_1(t,x)$ and $b_2(t,x)$ depend not only on the spatial variable x but also on the time variable t, we require the second condition (2.16) for the sake of the boundedness of the coefficients of the transformed system. In the same way as (2.5) we define the gauge transformation

$$\begin{aligned} v(t,x) &\longmapsto w(t,x) \equiv k_1(t,x)v(t,x), \\ k_1(t,x) &= e^{-p_1(t,x)}I, \\ p_1(t,x) &= \frac{1}{2}\int_0^x \operatorname{Im} b_1(t,y)dy. \end{aligned}$$

It is easy to see

$$e^{-M_1}\|v(t)\| \le \|w(t)\| \le e^{M_1}\|v(t)\|$$

$$\text{for any} \quad v(t,x) \in \Big(C([0,T]; L^2(\mathbb{R}))\Big)^2, t\in[0,T]. \tag{2.17}$$

Multiplying L_1 by $k_1(t,x)$, we have

$$\tilde{L}_1 v \equiv \Big(I\partial_t + i\tilde{H}_1(t)\Big)v = g(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}, \tag{2.18}$$

$$w(0,x) = w_0(x) \quad \text{in} \quad \mathbb{R}, \tag{2.19}$$

where

$$\begin{aligned}
g(t,x) &= k_1(t,x)f(t,x), \\
w_0(x) &= k_1(0,x)v_0(x), \\
\tilde{H}_1(t) &= A_1(\partial) - iB_1(t,x)\partial + C_1(t,x), \\
A_1(\partial) &= \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix}, \\
B_1(t,x) &= \begin{bmatrix} \operatorname{Re} b_1(t,x) & b_2(t,x) \\ \overline{b_2(t,x)} & \operatorname{Re} b_1(t,x) \end{bmatrix}, \\
C_1(t,x) &= \begin{bmatrix} c_{11}(t,x) & c_{12}(t,x) \\ c_{21}(t,x) & c_{22}(t,x) \end{bmatrix}, \\
c_{11}(t,x) &= -i\int_0^x \operatorname{Im}\partial_t b_1(t,y)dy \\
&\quad - \frac{i}{2}(\operatorname{Im} b_1(t,x))b_1(t,x) \\
&\quad + \frac{1}{4}(\operatorname{Im} b_1(t,x))^2 + \frac{1}{2}\operatorname{Im}\partial b_1(t,x), \\
c_{12}(t,x) &= -\frac{i}{2}b_2(t,x)\operatorname{Im} b_1(t,x), \\
c_{21}(t,x) &= -\frac{i}{2}\overline{b_2(t,x)}\operatorname{Im} b_1(t,x), \\
c_{22}(t,x) &= -i\int_0^x \operatorname{Im}\partial_t b_1(t,y)dy \\
&\quad - \frac{i}{2}(\operatorname{Im} b_1(t,x))\overline{b_1(t,x)} \\
&\quad - \frac{i}{2}(\operatorname{Im} b_1(t,x))\overline{b_1(t,x)} \\
&\quad - \frac{1}{4}(\operatorname{Im} b_1(t,x))^2 - \frac{1}{2}\operatorname{Im}\partial b_1(t,x).
\end{aligned}$$

It is easy to see that

Proposition 2.2.1 *We assume (2.15) and (2.16). Then (2.13)–(2.14) is H^s-well-posed for any $s \in \mathbb{R}$.*

To prove Proposition 2.2.1, we employ the duality type argument, which is based on the energy inequalities (see [23]).

Lemma 2.2.2 *There exists a constant $C_T > 0$ such that for any*

$$v \in \Big(C([0,T]; H^2(\mathbb{R})) \cap C([0,T]; L^2(\mathbb{R}))\Big)^2$$

and $t \in [0,T]$, we have

$$\|v(t)\| \le C_T\left(\|v(0)\| + \int_0^t \Big\|(L_1 v)(\tau)\Big\| d\tau\right), \tag{2.20}$$

$$\|v(t)\| \le C_T\left(\|v(T)\| + \int_t^T \Big\|(L_1^* v)(\tau)\Big\| d\tau\right), \tag{2.21}$$

where L_1^* is the formally adjoint operator of L_1, that is written as

$$\begin{aligned} L_1^* &= -I\partial_t - i\begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} \\ &- \begin{bmatrix} \overline{b_1(t,x)} & b_2(t,x) \\ \overline{b_2(t,x)} & b_1(t,x) \end{bmatrix}\partial \\ &+ \begin{bmatrix} \partial\overline{b_1(t,x)} & \partial b_2(t,x) \\ \partial\overline{b_2(t,x)} & \partial b_1(t,x) \end{bmatrix}. \end{aligned}$$

Proof. It sufficies to prove (2.20). We put $w(t,x) = k_1(t,x)v(t,x)$, $f(t,x) = (L_1 v)(t,x)$ and $g(t,x) = k_1(t,x)f(t,x)$. Then we have

$$\begin{aligned} w(t,x) &\in \Big(C([0,T]; H^2(\mathbb{R}))\Big)^2, \\ f(t,x),\ g(t,x) &\in \Big(C([0,T]; L^2(\mathbb{R}))\Big)^2. \end{aligned}$$

$w(t,x)$ satisfies

$$\begin{aligned} \tilde{L}_1 w &= g(t,x) && \text{in} \quad (0,T)\times\mathbb{R}, \\ w(0,x) &= k_1(0,x)w(0,x) && \text{in} \quad \mathbb{R}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d}{dt}\|w(t)\|^2 &= 2\mathrm{Re}\Big(I\partial_t w(t), w(t)\Big) \\ &= -2\mathrm{Re}\Big(i\tilde{H}_1(t)w(t), w(t)\Big) + 2\mathrm{Re}\Big(g(t), w(t)\Big) \\ &= -2\mathrm{Re}\Big(a_1(\partial)w(t), w(t)\Big) + 2\mathrm{Re}\Big(B_1(t,x)\partial w(t), w(t)\Big) \\ &+ 2\mathrm{Re}\Big(iC_1(t,x)w(t), w(t)\Big) + 2\mathrm{Re}\Big(g(t), w(t)\Big) \\ &\le 2\mathrm{Re}\Big(B_1(t,x)\partial w(t), w(t)\Big) + CB_{b_1}(t)\|w(t)\|^2 + 2\|w(t)\|\|g(t)\|. \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} B_{b_1}(t) &= \sup_{x\in\mathbb{R}}\left|\int_0^x \mathrm{Im}\,\partial_t b_1(t,y)dy\right| \\ &+ \sum_{j=1,2}\sum_{l=0,1}\Big(\sup_{x\in\mathbb{R}}|\partial^l b_j(t,x)| + \sup_{x\in\mathbb{R}}|\partial^l b_j(t,x)|^2\Big). \end{aligned}$$

We remark here that $B_1(t,x)$ is a hermitian matrix, that is $\overline{B_1(t,x)} = {}^tB_1(t,x)$. The integration by parts yields

$$\begin{aligned} & 2\mathrm{Re}\Big(B_1(t,x)\partial w(t), w(t)\Big) \\ &= \int_{-\infty}^{+\infty} {}^t\overline{w(t)}B_1(t,x)\partial w(t)dx + \int_{-\infty}^{+\infty} {}^tw(t)\overline{B_1(t,x)}\,\overline{\partial w(t)}dx \\ &= \int_{-\infty}^{+\infty} \partial^t w(t)^t B_1(t,x)\overline{w(t)}dx + \int_{-\infty}^{+\infty} {}^tw(t)\overline{B_1(t,x)\partial w(t)}dx \\ &= \int_{-\infty}^{+\infty} \partial^t w(t)^t B_1(t,x)\overline{w(t)}dx + \int_{-\infty}^{+\infty} {}^tw(t)^tB_1(t,x)\overline{\partial w(t)}dx \\ &= -\int_{-\infty}^{+\infty} {}^tw(t)^tB_1(t,x)\overline{\partial w(t)}dx + \int_{-\infty}^{+\infty} {}^tw(t)^tB_1(t,x)\overline{\partial w(t)}dx \\ &+ -\int_{-\infty}^{+\infty} {}^tw(t)\partial^t B_1(t,x)\overline{w(t)}dx \\ &= -\int_{-\infty}^{+\infty} {}^tw(t)\partial^t B_1(t,x)\overline{w(t)}dx \\ &\le CB_{b_1}(t)\|w(t)\|^2 \end{aligned} \tag{2.23}$$

Substitution of (2.23) into (2.22) gives

$$\frac{d}{dt}\|w(t)\| \le CB_{b_1}(t)\|w(t)\| + \|g(t)\|. \tag{2.24}$$

(2.24) plays an essential role when we solve semilinear equations. Integrating (2.24) on $[0,t]$, we get

$$\|w(t)\| \le e^{Ct}\|w(0)\| + \int_0^t e^{C(t-\tau)}\|g(\tau)\|d\tau \quad \text{for} \quad t \in [0,T]. \tag{2.25}$$

Combining (2.17) and (2.25), we obtain (2.20). ∎

We can prove Proposition 2.2.1 by Lemma 2.2.2. The rigourous proof of it is omitted here. See the proof of Proposition 2.3.2.

2.3 Linear estimates in general space dimensions

In this section we study the initial value problem for the following linear Schrödinger-type systems

$$Lv \equiv \Big(I\partial_t + iH(t)\Big)v = f(t,x) \quad \text{in} \quad (0,T)\times\mathbb{R}^N, \tag{2.26}$$

$$v(0,x) = v_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{2.27}$$

where $v = v(t,x)$ is $\mathbb{C}^2$-valued, I is the 2×2 identity matrix and the operator $H(t) = h(t,x,D)$ is defined by

$$\begin{aligned} h(t,x,\xi) &= a(\xi) + b(t,x,\xi), \qquad a(\xi) = \begin{bmatrix} |\xi|^2 & 0 \\ 0 & -|\xi|^2 \end{bmatrix}, \\ b(t,x,\xi) &= \begin{bmatrix} b_{11}(t,x,\xi) & b_{12}(t,x,\xi) \\ b_{21}(t,x,\xi) & b_{22}(t,x,\xi) \end{bmatrix} = \sum_{j=1}^{N} \begin{bmatrix} b_{11j}(t,x) & b_{12j}(t,x) \\ b_{21j}(t,x) & b_{22j}(t,x) \end{bmatrix} \xi_j, \\ b_{mnj}(t,x) &\in C^1\Big([0,T];\mathcal{B}^\infty\Big), \quad m,n = 1,2 \text{ and } j = 1,\cdots,N. \end{aligned}$$

For the sake of the convenience, we put

$$b^{\text{diag}}(t,x,\xi) = \begin{bmatrix} b_{11}(t,x,\xi) & 0 \\ 0 & b_{22}(t,x,\xi) \end{bmatrix}, \quad b^{\text{off}}(t,x,\xi) = \begin{bmatrix} 0 & b_{12}(t,x,\xi) \\ b_{21}(t,x,\xi) & 0 \end{bmatrix}.$$

When we apply the linear estimates in this section to (1.1)–(1.2), we see v as $v = {}^t(u,\bar{u})$. For H^s-well-posedness of (2.26)–(2.27), we assume the Doi-type condition on $b^{\text{diag}}(t,x,\xi)$, that is to say, there exist functions

$$\varphi_j(t,s) \in C\Big([0,T];\mathcal{B}^\infty(\mathbb{R})\Big) \cap C^1\Big([0,T];L^1(\mathbb{R})\Big), \quad j = 1,\cdots,N$$

such that

$$\Big|\text{Im}\, b_{nnj}(t,x)\Big| \le \varphi_j(t,x_j) \quad \text{for} \quad (t,x)\in[0,T]\times\mathbb{R}^N, \quad n = 1,2, \quad j = 1,\cdots,N. \tag{2.28}$$

Since our analysis is based on the symbolic calculus for pseudo-differential operators, we quote the minimum fundamentals on them from H. Kumano-go's textbook [33]. Let $m\in\mathbb{R}$ and let S^m be a class of symbols defined by

$$S^m = S^m(\mathbb{R}^N\times\mathbb{R}^N) = \Big\{p(x,\xi)\in C^\infty(\mathbb{R}^N\times\mathbb{R}^N) \;\Big|\; |p|_l^{(m)} < +\infty,\ l\in\mathbb{Z}_+\Big\},$$

where

$$|p|_l^{(m)} = \sup_{\substack{|\alpha+\beta|\le l \\ x,\xi\in\mathbb{R}^N}} \Big|p_{(\beta)}^{(\alpha)}(x,\xi)\langle\xi\rangle^{-m+|\alpha|}\Big|, \quad p_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D^\beta p(x,\xi), \quad \alpha,\beta\in(\mathbb{Z}_+)^N.$$

S^m is a Fréchet space with respect to the family of seminorms $|\cdot|_l^{(m)}$, $l\in\mathbb{Z}_+$. If $p(x,\xi)\in S^m$ is given, then $p(x,\xi)$ defines the operator P by

$$\begin{aligned} Pu(x) &= p(x,D)u(x) \\ &= (2\pi)^{-N}\iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{i(x-y)\cdot\xi}p(x,\xi)u(y)\,dyd\xi \\ &= (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi, \quad \text{for} \quad u\in\mathcal{S}. \end{aligned}$$

Conversely, if the operator P is given, then its symbol $\sigma(P)(x,\xi)$ is calculated by

$$\sigma(P)(x,\xi) = e^{-ix\cdot\xi}P(e^{ix\cdot\xi}).$$

The properties of pseudo-differential operators are the following.

Lemma 2.3.1 (1) *Let $p(x,\xi)\in S^m$. Then $P = p(x,D)\in\mathcal{L}(H^{s+m},H^s)$ for any $s\in\mathbb{R}$ and there exist $l_m = l(m,s,N)\in\mathbb{N}$ and $C = C(s,N) > 0$ such that*

$$\|Pu\|_s \le C|p|_{l_m}^{(m)}\|u\|_{s+m} \quad \textit{for} \quad u\in H^{s+m}.$$

(2) *Let $p(x,\xi)\in S^m$ and $p'(x,\xi)\in S^{m'}$, and let $\alpha,\alpha',\beta,\beta'\in(\mathbb{Z}_+)^N$. We define*

$$r_\theta(x,\xi) = (2\pi)^{-N}\iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{-iy\cdot\eta}p_{(\beta)}^{(\alpha)}(x,\xi+\theta\eta)p'^{(\alpha')}_{(\beta')}(x+y,\xi)\,dyd\eta \quad \textit{for} \quad \theta\in[0,1].$$

Then $\{r_\theta\}_{\theta\in[0,1]}$ is bounded in $S^{m+m'-|\alpha+\alpha'|}$ and for any $l\in\mathbb{Z}_+$, there exist $l'\in\mathbb{Z}_+$ and $C_l > 0$, which are independent of $\theta\in[0,1]$, such that

$$|r_\theta|_l^{(m+m'-|\alpha+\alpha'|)} \le C_l|p|_{l'}^{(m)}|p'|_{l'}^{(m')}.$$

(3) *Let $p(x,\xi)\in S^m$ and $p'(x,\xi)\in S^{m'}$, and let $P = p(x,D)$ and $P' = p'(x,D)$. Then $\sigma(PP')(x,\xi)\in S^{m+m'}$ and*

$$\sigma(PP')(x,\xi) = (2\pi)^{-N}\iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{-iy\cdot\eta}p(x,\xi+\eta)p'(x+y,\xi)\,dyd\eta.$$

(4) (**The Sharp Gårding inequality**) *For any $p(x,\xi)\in S^1$ satisfying*

$$Re\,p(x,\xi)\ge 0 \quad \textit{for} \quad |\xi|\ge R\ge 0 \quad \textit{with some } R\ge 0,$$

there exist $l\in\mathbb{N}$ and $C = C(R,N) > 0$ such that

$$Re(Pu,u)_{L^2} \ge -C|p|_l^{(1)}\|u\|_{L^2}^2 \quad \textit{for} \quad u\in\mathcal{S}.$$

In this section we prove

Proposition 2.3.2 *We assume* (2.3). *Then the initial value problem* (2.26)–(2.27) *is H^s-well-posed for any $s \in \mathbb{R}$, that is to say, for any $v_0(x) \in (H^s)^2$ and for any $f(t,x) \in \left(L^1_{loc}\left(0,T;H^s\right)\right)^2$, (2.26)–(2.27) possesses a unique solution $v \in \left(C\left([0,T];H^s\right)\right)^2$.*

We have only to prove Proposition 2.3.2 with $s = 0$. Our strategy divides into two steps. At the first step we diagonalize the operator $H(t)$ modulo bounded operators. Roughly speaking, its symbol

$$h(t,x,\xi) = \begin{bmatrix} |\xi|^2 + b_{11}(t,x,\xi) & b_{12}(t,x,\xi) \\ b_{21}(t,x,\xi) & -|\xi|^2 + b_{22}(t,x,\xi) \end{bmatrix}$$

has two distinct eigen-values provided that $|\xi|$ is sufficiently large. Thus we can easily diagonalize $h(t,x,\xi)$ and therefore this system becomes a couple of single Schrödinger-type equations essentially. At the second step we apply S. Doi's method ([11], [12]) to the diagonalized system.

To prove Proposition 2.3.2, we introduce some pseudo-differential operators. The diagonalization is carried out by

$$\Lambda(t) = I + \tilde{\Lambda}(t), \quad \Lambda'(t) = I - \tilde{\Lambda}(t), \quad \tilde{\Lambda}(t) = \tilde{\lambda}(t,x,D),$$

$$\tilde{\lambda}(t,x,\xi) = \frac{1}{2}\sum_{j=1}^{N} \begin{bmatrix} 0 & b_{12j}(t,x) \\ -b_{21j}(t,x) & 0 \end{bmatrix} \xi_j \langle\xi\rangle^{-2}.$$

The loss of derivatives is resolved by

$$\begin{aligned} K(t) &= k(t,x,D), & K'(t) &= k'(t,x,D), \\ k(t,x,\xi) &= \begin{bmatrix} e^{-p(t,x,\xi)} & 0 \\ 0 & e^{p(t,x,\xi)} \end{bmatrix}, & k'(t,x,\xi) &= \begin{bmatrix} e^{p(t,x,\xi)} & 0 \\ 0 & k_0(t,x,\xi) \end{bmatrix}, \end{aligned}$$

$$p(t,x,\xi) = \sum_{j=1}^{N} \int_0^{x_j} \phi_j(t,s)ds\,\xi_j\langle\xi_j\rangle^{-1}.$$

It is convenient to use the following notations

$$\begin{aligned} B_K(t) &= |e^{-p(t)}|_l^{(0)} + |e^{p(t)}|_l^{(0)} \geq 2, \\ B_b(t) &= \sum_{m,n=1,2}\sum_{j=1}^{N}\sum_{|\alpha|\leq l}\left(\sup_{x\in\mathbb{R}^N}|\partial^\alpha b_{mnj}(t,x)| + \sup_{x\in\mathbb{R}^N}|\partial_t\partial^\alpha b_{mnj}(t,x)|\right), \\ B_\phi^0(t) &= \sum_{j=1}^{N}\int_{-\infty}^{+\infty}\phi_j(t,x_j)dx_j, \quad B_\phi^1(t) = \sum_{j=1}^{N}\sup_{x_j\in\mathbb{R}}\left|\int_0^{x_j}\partial_t\phi_j(t,y_j)dy_j\right|, \\ B_\phi^\infty(t) &= \sum_{j=1}^{N}\sum_{k=0}^{l}\sup_{x_j\in\mathbb{R}}|\partial_j^k\phi_j(t,x_j)|. \end{aligned}$$

where $l \in \mathbb{N}$ is large enough to be used in this section. To eliminate the loss of derivatives in (2.26), we use the map $v \longmapsto K(t)\Lambda(t)v$. Before we use this map, we verify that this is automorphic on $(L)^2$ in a sense.

Lemma 2.3.3 *There exists a constant $C > 0$ such that*

$$\begin{aligned} \|v\| &\leq CB_K(t)\left(1 + B_b(t)\right)^2\left(1 + B_\phi^0(t)B_\phi^\infty(t)\right)N_t(v), && (2.29) \\ N_t(v) &\leq CB_K(t)\left(1 + B_b(t)\right)\|v\| && (2.30) \end{aligned}$$

for $v \in (L^2)^2$ and $t \in [0,T]$, where $N_t(v) = \|K(t)\Lambda(t)v\| + \|v\|_{-1}$.

Proof. Simple calculation gives

$$I = \Lambda'(t)K'(t)K(t)\Lambda(t) + \tilde{\Lambda}(t)^2 + \Lambda'(t)R_0(t)\Lambda(t),$$

$$\begin{aligned} \sigma\left(R_0(t)\right)(x,\xi) &= \sigma\left(I - K'(t)K(t)\right)(x,\xi) \\ &= -i\sum_{j=1}^{N}\int_0^{x_j}\phi_j(t,s)ds\,\xi_j\langle\xi_j\rangle^{-1} \\ &\times\ (2\pi)^{-N}\int_0^1\iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{-iy\cdot\eta}\langle\xi_j + \theta\eta_j\rangle^{-3}k'(t,x,\xi+\theta\eta) \\ &\qquad\times\phi_j(t,x_j+y_j)k(t,x+y,\xi)dyd\eta d\theta. \end{aligned}$$

Then it is easy to see (2.29) and (2.30). ∎

We diagonalize L modulo bounded operators by $\Lambda(t)$ and $\Lambda'(t)$. Here we note $I = \Lambda'(t)\Lambda(t) + \tilde{\Lambda}(t)^2$ and $\sigma\left(\tilde{\Lambda}(t)^2\right)(x,\xi) \in (S^{-2})^{2\times 2}$. Operating $\Lambda(t)$ on L we have

$$\begin{aligned} \Lambda(t)Lv &= I\partial_t\left(\Lambda(t)v\right) - \left(\partial_t\tilde{\Lambda}(t)\right)v + i\Lambda(t)H(t)\Lambda'(t)\Lambda(t)v + iR_1(t)v, && (2.31) \\ \sigma\left(\partial_t\tilde{\Lambda}(t)\right)(x,\xi) &= \partial_t\tilde{\lambda}(t,x,\xi) = \frac{1}{2}\sum_{j=1}^{N}\begin{bmatrix} 0 & \partial_t b_{12j}(t,x) \\ -\partial_t b_{21j}(t,x) & 0 \end{bmatrix}\xi_j\langle\xi\rangle^{-2}, \\ R_1(t) &= \Lambda(t)H(t)\tilde{\Lambda}(t)^2. \end{aligned}$$

The simple calculation yields

$$\begin{aligned} \Lambda(t)H(t)\Lambda'(t) &= \left(I + \tilde{\Lambda}(t)\right)\left(a(D) + b(t,x,D)\right)\left(I - \tilde{\Lambda}(t)\right) \\ &= a(D) + b(t,x,D) + \tilde{\Lambda}(t)a(D) - a(D)\tilde{\Lambda}(t) + R_2(t), && (2.32) \\ R_2(t) &= \tilde{\Lambda}(t)b(t,x,D) - b(t,x,D)\tilde{\Lambda}(t) - \tilde{\Lambda}(t)H(t)\tilde{\Lambda}(t). \end{aligned}$$

Noting $\langle D\rangle^{-2}|D|^2 = 1-\langle D\rangle^{-2}$, we get

$$\tilde{A}(t)a(D) = -\frac{1}{2}b^{\text{off}}(t,x,D)+\frac{1}{2}b^{\text{off}}(t,x,D)\langle D\rangle^{-2}, \tag{2.33}$$

$$-a(D)\tilde{A}(t) = -\frac{1}{2}b^{\text{off}}(t,x,D)+\frac{1}{2}b^{\text{off}}(t,x,D)\langle D\rangle^{-2}+R_3(t), \tag{2.34}$$

$$\sigma\Big(R_3(t)\Big)(x,\xi) = -(2\pi)^{-N}\sum_{j=1}^{N}\int_0^1\iint_{\mathbb{R}^N\times\mathbb{R}^N} e^{-iy\cdot\eta}a^{(j)}(\xi+\theta\eta)\tilde{\lambda}_{(j)}(t,x+y,\xi)dyd\eta d\theta.$$

Substituting (2.33) and (2.34) into (2.32), we have

$$\begin{aligned} A(t)H(t)A'(t) &= a(D)+b^{\text{diag}}(t,x,D)+R_2(t)+R_3(t)+R_4(t), \\ R_4(t) &= b^{\text{off}}(t,x,D)\langle D\rangle^{-2}. \end{aligned} \tag{2.35}$$

Combining (2.31) and (2.35), we obtain

$$\begin{aligned} A(t)Lv &= I\partial_t\Big(A(t)v\Big)+i\Big(a(D)+b^{\text{diag}}(t,x,D)\Big)\Big(A(t)v\Big)+R_5(t), \\ R_5(t) &= -\Big(\partial_t\tilde{A}(t)\Big)+iR_1(t)+i\Big(R_2(t)+R_3(t)R_4(t)\Big)A(t). \end{aligned}$$

It is easy to see

$$\sigma\Big(R_5(t)\Big)(x,\xi)\in(S^0)^{2\times 2},\quad \|R_5(t)\|_{\mathcal{L}\left((L^2)^2\right)}\le CB_b(t)\Big(1+B_b(t)\Big)^3. \tag{2.36}$$

Our diagonalization is completed.

We resolve $\operatorname{Im}b^{\text{diag}}(t,x,\xi)$ by $K(t)$. Roughly speaking, $\sigma\Big(i[K(t),a(D)]K'(t)\Big)(x,\xi)$ gives a elliptic term which is stronger than $\operatorname{Im}b^{\text{diag}}(t,x,\xi)$. Since $k(t,x,\xi)$, $a(\xi)$ and $b^{\text{diag}}(t,x,\xi)$ are diagonal matrices, we can deal with them as if they were scalar-valued symbols. Operating $K(t)$ on $A(t)Lv$, we have

$$\begin{aligned} K(t)A(t)Lv &= I\partial_t\Big(K(t)A(t)v\Big)-\Big(\partial_tK(t)\Big)\Big(A(t)v\Big) \\ &+ i\Big(a(D)+b^{\text{diag}}(t,x,D)\Big)\Big(K(t)A(t)v\Big)+i[K(t),a(D)]\Big(A(t)v\Big) \\ &+ R_6(t)A(t)v+K(t)R_5(t)v, \end{aligned} \tag{2.37}$$

$$\sigma\Big(\partial_tK(t)\Big)(x,\xi)=\partial_tk(t,x,\xi)=\sum_{j=1}^{N}\int_0^{x_j}\phi_j(t,s)ds\xi_j\langle\xi_j\rangle^{-1}\tilde{I}k(t,x,\xi),$$

$$\tilde{I}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad \|\partial_tK(t)\|_{\mathcal{L}\left((L^2)^2\right)}\le CB^1_\phi(t)B_K(t), \tag{2.38}$$

$$R_6(t)=i[K(t),b^{\text{diag}}(t,x,D)],\quad \|R_6(t)\|_{\mathcal{L}\left((L^2)^2\right)}\le CB_b(t)B^\infty_\phi(t)B_K(t). \tag{2.39}$$

Now we will calculate the commutator

$$i[K(t),a(D)]=\begin{bmatrix} i[e^{-p(t,x,D)},|D|^2] & 0\\ 0 & -i[e^{p(t,x,D)},|D|^2]\end{bmatrix}.$$

The simple calculation gives

$$\begin{aligned} &\sigma\Big(i[e^{-p(t,x,D)},|D|^2]\Big)(x,\xi) \\ &= 2\sum_{j=1}^{N}\phi_j(t,x_j)\xi_j^2\langle\xi_j\rangle^{-1}e^{-p(t,x,\xi)}+r_7(t,x,\xi) \\ &= 2\sum_{j=1}^{N}\phi_j(t,x_j)\sigma\Big(D_j^2\langle D_j\rangle^{-1}e^{-p(t,x,D)}\Big)(x,\xi)+r_7(t,x,\xi)+r_8(t,x,\xi), \end{aligned}$$

$$\begin{aligned} r_7(t,x,\xi) &= i\sum_{j=1}^{N}\Big(-\partial_j\phi_j(t,x_j)\xi_j\langle\xi_j\rangle^{-1}+\phi_j(t,x_j)^2\xi_j^2\langle\xi_j\rangle^{-2}\Big)e^{-p(t,x,\xi)}, \\ r_8(t,x,\xi) &= -4i\phi_j(t,x_j)\xi_j\langle\xi_j\rangle^{-1} \\ &\quad\times(2\pi)^{-N}\int_0^1\iint_{\mathbb{R}^N\times\mathbb{R}^N}e^{-iy\cdot\eta}(\xi_j+\theta\eta_j)\langle\xi_j+\theta\eta_j\rangle^{-3} \\ &\quad\quad\times\phi_j(t,x_j+y_j)e^{-p(t,x+y,\xi)}dyd\eta d\theta. \end{aligned}$$

In the same way we get

$$\begin{aligned} &\sigma\Big(-i[e^{p(t,x,D)},|D|^2]\Big)(x,\xi) \\ &= 2\sum_{j=1}^{N}\phi_j(t,x_j)\sigma\Big(D_j^2\langle D_j\rangle^{-1}e^{p(t,x,D)}\Big)(x,\xi)+r_7'(t,x,\xi)+r_8'(t,x,\xi), \end{aligned}$$

where $r_7'(t,x,\xi)$ and $r_8'(t,x,\xi)$ are the similar symbols to $r_7(t,x,\xi)$ and $r_8(t,x,\xi)$ respectively. Then we have

$$i[K(t),a(D)]=2\sum_{j=1}^{N}\phi_j(t,x_j)D_j^2\langle D_j\rangle^{-1}K(t)+R_9(t), \tag{2.40}$$

$$\|R_9(t)\|_{\mathcal{L}\left((L^2)^2\right)}\le CB^\infty_\phi(t)\Big(1+B^\infty_\phi(t)\Big)B_K(t). \tag{2.41}$$

Substituting (2.40) into (2.37), we obtain

$$\begin{aligned} K(t)A(t)Lv &= \Big(I\partial_t+ia(D)+q(t,x,D)\Big)K(t)A(t)v+R_{10}(t)v, \\ q(t,x,\xi) &= \begin{bmatrix} q_1(t,x,\xi) & 0\\ 0 & q_2(t,x,\xi)\end{bmatrix} \end{aligned} \tag{2.42}$$

$$
\begin{aligned}
&= 2I\sum_{j=1}^{N}\phi_j(t,x_j)\xi_j^2\langle\xi_j\rangle^{-1} + ib^{\mathrm{diag}}(t,x,\xi)\\
&= \sum_{j=1}^{N}\begin{bmatrix} 2\phi_j(t,x_j)\xi_j^2\langle\xi_j\rangle^{-1} + ib_{11j}(t,x)\xi_j & 0 \\ 0 & 2\phi_j(t,x_j)\xi_j^2\langle\xi_j\rangle^{-1} + ib_{22j}(t,x)\xi_j \end{bmatrix},\\
R_{10}(t) &= -\Big(\partial_t K(t)\Big) + R_6(t)\Lambda(t) + R_9(t)\Lambda(t) + K(t)R_5(t).
\end{aligned}
$$

With (2.36), (2.38), (2.39) and (2.41), we have

$$
\|R_{10}(t)\|_{\mathcal{L}\left((L^2)^2\right)} \le C\Big(B_b(t) + B_\phi^1(t) + B_\phi^\infty(t)\Big)\Big(1 + B_\phi^\infty(t)\Big)\Big(1 + B_b(t)\Big)^3 B_K(t). \tag{2.43}
$$

Our deformation of the operator L is finished.

To complete the proof of Proposition 2.3.2, we have only to show the following energy inequalities. More precisely, Proposition 2.3.2 follows from the duality type arguments for the forward and the backward initial value problems. The energy inequalities play an essential role in these arguments.

Lemma 2.3.4 *There exists a constant $C_T > 0$ such that*

$$
\|v(t)\| \le C_T\left(\|v(0)\| + \int_0^t \|(Lv)(\tau)\|d\tau\right), \tag{2.44}
$$

$$
\|v(t)\| \le C_T\left(\|v(T)\| + \int_t^T \|(L^*v)(\tau)\|d\tau\right), \tag{2.45}
$$

$$
\textit{for} \quad v \in \Big(C([0,T];H^2)\cap C^1([0,T];L^2)\Big)^2, \quad t\in[0,T].
$$

Here L^* is the formally adjoint operator of L, that is

$$
\begin{aligned}
L^* &= -I\partial_t - i\Big(a(D) + b(t,x,D)^*\Big),\\
\sigma\Big(b(t,x,D)^*\Big)(x,\xi) &= \overline{b(t,x,\xi)} - i\sum_{j=1}^{N}\begin{bmatrix} \partial_j\overline{b_{11j}(t,x)} & \partial_j\overline{b_{12j}(t,x)} \\ \partial_j\overline{b_{21j}(t,x)} & \partial_j\overline{b_{22j}(t,x)}\end{bmatrix}.
\end{aligned}
$$

Proof. We put $w = K(t)\Lambda(t)v$, $f(t,x) = (Lv)(t,x)$ and $g = K(t)\Lambda(t)f$. Then (2.42) implies

$$
\Big(I\partial_t + ia(D) + q(t,x,D)\Big)w + R_{10}(t)v = g.
$$

We have

$$
\begin{aligned}
\frac{d}{dt}\|w(t)\|^2 &= 2\mathrm{Re}\Big(I\partial_t w(t), w(t)\Big)\\
&= -2\mathrm{Re}\Big(ia(D)w(t), w(t)\Big) - 2\mathrm{Re}\Big(q(t,x,D)w(t), w(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -2\mathrm{Re}\Big(R_{10}(t)v(t), w(t)\Big) + 2\mathrm{Re}\Big(g(t), w(t)\Big)\\
&\le -2\mathrm{Re}\Big(q(t,x,D)w(t), w(t)\Big)\\
&\quad +2\Big(\|R_{10}(t)\|_{\mathcal{L}\left((L^2)^2\right)}\|v(t)\| + \|g(t)\|\Big)\|w(t)\|.
\end{aligned} \tag{2.46}
$$

Here we remark that (2.28) yields

$$
\mathrm{Re}\, q_n(t,x,\xi) \ge 0 \qquad \text{for} \quad (t,x)\in[0,T]\times\mathbb{R}^N, \quad |\xi|\ge 1, \quad n = 1,2.
$$

Then the sharp Gårding inequality implies

$$
-2\mathrm{Re}\Big(q(t,x,D)w(t), w(t)\Big) \le C\Big(B_b(t) + B_\phi^\infty(t)\Big)\|w(t)\|^2. \tag{2.47}
$$

Substituting (2.29), (2.43) and (2.47) into (2.46), we get

$$
\frac{d}{dt}\|w(t)\| \le C\Big(B_b(t) + B_\phi^1(t) + B_\phi^\infty(t)\Big)B_{\mathrm{etc}}(t)N_t\Big(v(t)\Big) + \|g(t)\|, \tag{2.48}
$$

$$
B_{\mathrm{etc}}(t) = \Big(1 + B_b(t)\Big)^5\Big(1 + B_\phi^0(t)\Big)\Big(1 + B_\phi^\infty(t)\Big)^2 B_K(t)^2.
$$

On the other hand, we have

$$
\frac{d}{dt}\|v(t)\|_{-1} \le CB_b(t)B_{\mathrm{etc}}(t)N_t\Big(v(t)\Big) + \|f(t)\|_{-1}. \tag{2.49}
$$

Combining (2.48) and (2.49), we obtain

$$
\frac{d}{dt}N_t\Big(v(t)\Big) \le C\Big(B_b(t) + B_\phi^1(t) + B_\phi^\infty(t)\Big)B_{\mathrm{etc}}(t)N_t\Big(v(t)\Big) + N_t\Big(f(t)\Big). \tag{2.50}
$$

(2.29), (2.30) and (2.50) implies (2.44). In the same way we can get (2.45). This completes the proof of Lemma 2.3.4. ∎

We will prove Proposition 2.3.2. As we mentioned before, we prove it by using Lemma 2.3.4. Basically, we follow L. Hörmander's text book [23, Section 23.1], where he applied such argument to proving H^s-well-posedness of the Cauchy problem for the first order pseudo-differential equations. Of course, such argument is also applicable to the Cauchy problem for the strictly hyperbolic systems and the initial value problem for our systems.

Proof of Proposition 2.3.2.

Uniqueness. Let $v \in \Big(C([0,T];H^s)\Big)^2$ be a solution to (2.26)-(2.27) with $v_0 = 0$ and $f = 0$. Because

$$
\begin{aligned}
Lv(=0) &\in \Big(C([0,T];H^{s-2})\Big)^2,\\
H(t)v &\in \Big(C([0,T];H^{s-2})\Big)^2,
\end{aligned}
$$

we obtain

$$I\partial_t v\big(= -iH(t)\big) \in \Big(C([0,T];H^{s-2})\Big)^2.$$

Then we can use the energy ineuality (2.44) and we get $\|v(t)\|_{s-2} = 0$ for any $t \in [0,T]$.

Existence. If there exists a solution v to the initial value problem (2.26)–(2.27), then we have

$$\int_0^T (Lv, w)dt = \int_0^T (f,w)dt$$

$$\text{for any} \quad w \in \Big(C_0^\infty([0,T) \times \mathbb{R}^N)\Big)^2.$$

Using the integration by parts, we get

$$\int_0^T (v, L^*w)dt = \int_0^T (f,w)dt + (v_0, w(0)). \tag{2.51}$$

We remark here that the energy inequality (2.45) implies

$$\sup_{t\in[0,T]} \|w(t)\|_{-s} \le C_T \int_0^T \|L^*w(t)\|_{-s}dt \tag{2.52}$$

$$\text{for any} \quad w \in \Big(C_0^\infty([0,T) \times \mathbb{R}^N)\Big)^2.$$

Applying the energy inequality (2.52) to (2.51), we obtain

$$\begin{aligned} &\left|\int_0^T (f,w)dt + (v_0, w(0))\right| \\ \le\ & \left(\int_0^T \|f(t)\|_s dt + \|v_0\|_s\right) \sup_{t\in[0,T]} \|w(t)\|_{-s} \\ \le\ & C_T \left(\int_0^T \|f(t)\|_s dt + \|v_0\|_s\right) \int_0^T \|L^*w(t)\|_{-s}dt \\ =\ & C_T\left(\|f\|_{(L^1(0,T;H^s))^2} + \|v_0\|_s\right)\|L^*w\|_{(L^1(0,T;H^{-s}))^2}. \end{aligned} \tag{2.53}$$

Now we define the vector space X_T^s by

$$X_T^s = \left\{ L^*w \;\middle|\; w \in \Big(C_0^\infty([0,T) \times \mathbb{R}^N)\Big)^2 \right\}.$$

Clearly, X_T^s is a subspace of $\Big(L^1(0,T;H^{-s})\Big)^2$.

For given $f \in \Big(L^1(0,T;H^s)\Big)^2$ and $v_0 \in (H^s)^2$, we see

$$\int_0^T (f,w)dt + (v_0, w(0))$$

as an anti-linear form on X_T^s. The estimate (2.53) asserts that this is a bounded anti-linear form on X_T^s under the $\Big(L^1(0,T;H^{-s})\Big)^2$-topology. Then the Hahn–Banach theorem (see, e.g., [40, Theorem III.6]) implies that there exists a bounded anti-linear form Ψ defined on $\Big(L^1(0,T;H^{-s})\Big)^2$ such that

$$\begin{aligned} \left|\langle \Psi, \overline{z}\rangle\right| &\le C_T\left(\int_0^T \|f(t)\|_s dt + \|v_0\|_s\right)\int_0^T \|z(t)\|_{-s}dt \\ &= C_T\left(\|f\|_{(L^1(0,T;H^s))^2} + \|v_0\|_s\right)\|z\|_{(L^1(0,T;H^{-s}))^2} \end{aligned}$$

for any $z \in \Big(L^1(0,T;H^{-s})\Big)^2$, and

$$\langle \Psi, \overline{L^*w}\rangle = \int_0^T (f,w)dt + (v_0, w(0))$$

for any $w \in \Big(C_0^\infty([0,T) \times \mathbb{R}^N)\Big)^2$. Then the theorem of the reprezentation of the duality implies that there exists $v \in \Big(L^\infty(0,T;H^s)\Big)^2$ such that

$$\langle \Psi, \overline{z}\rangle = \int_0^T (v,z)dt$$

for any $z \in \Big(L^1(0,T;H^{-s})\Big)^2$. In particular,

$$\int_0^T (v, L^*w)dt = \int_0^T (f,w)dt + (v_0, w(0))$$

for any $w \in \Big(C_0^\infty([0,T) \times \mathbb{R}^N)\Big)^2$. This means that v is a solution to

$$Lv = f \quad \text{in} \quad \Big(\mathcal{D}'\big((0,T) \times \mathbb{R}^N\big),\Big)^2 \tag{2.54}$$

$$v(0,\cdot) = v_0 \quad \text{in} \quad \Big(\mathcal{D}'(\mathbb{R}^N)\Big)^2. \tag{2.55}$$

Regularity. We prove the continuity of v in the time-variable t. We take sequences

$$\{v_0^n\}_{n\ge1} \subset \Big(C_0^\infty(\mathbb{R}^N)\Big)^2, \qquad \{f^n\}_{n\ge1} \subset \Big(C^\infty([0,T];\mathcal{S}(\mathbb{R}^N))\Big)^2,$$

which satisfy

$$\begin{aligned} v_0^n &\longrightarrow v_0 \quad \text{in} \quad \Big(H^s\Big)^2 \quad \text{as} \quad n\to\infty, \\ f^n &\longrightarrow f \quad \text{in} \quad \Big(L^1(0,T;H^s)\Big)^2 \quad \text{as} \quad n\to\infty. \end{aligned}$$

We suppose that $v \in \left(L^\infty(0,T;H^s)\right)$ is a unique solution to (2.54)–(2.55). Let v^n be a unique solution to

$$
\begin{aligned}
Lv^n &= f^n \quad \text{in} \quad \left(\mathcal{D}'\left((0,T)\times\mathbb{R}^N\right),\right)^2 \\
v^n(0,\cdot) &= v_0^n \quad \text{in} \quad \left(\mathcal{D}'(\mathbb{R}^N)\right)^2.
\end{aligned}
$$

Clearly, each v^n belongs to $\left(C^\infty([0,T];H^\infty)\right)^2$. The energy inequality (2.44) yields

$$
\|v^n(t)-v^m(t)\|_s \le C_T\left(\|v_0^n - v_0^m\|_s + \int_0^t \|f^n(\tau)-f^m(\tau)\|_s d\tau\right).
$$

Then, it follows that $\{v^n\}_{n\ge1}$ is a Cauchy sequence in $\left(C([0,T];H^s)\right)^2$. The uniqueness of the solution implies that $v \in \left(C([0,T];H^s)\right)^2$ and

$$
v^n \longrightarrow v \quad \text{in} \quad \left(C([0,T];H^s)\right)^2 \quad \text{as} \quad n\to\infty.
$$

This complets the proof of Proposition 2.3.2. ∎

Remark 2.3.1 When we apply Proposition 2.3.2 or Lemma 2.3.4 to (1.1)–(1.2), we take $\phi_j(t,x_j)$ satisfying

$$
\sup_{\check{x}_j\in\mathbb{R}^{N-1}}\left|\frac{\partial F}{\partial q_j}(u,\nabla u)(t,x)\right| \le \phi_j(t,x_j), \quad j=1,\cdots,N. \tag{2.56}
$$

Then we choose $\phi_j(t,x_j)$ as

$$
\phi_j(t,x_j) = M\int_{\mathbb{R}^{N-1}}\left|\langle D\rangle^{[(N-1)/2]+2}u(t,x)\right|^2 d\check{x}_j \tag{2.57}
$$

or

$$
\phi_j(t,x_j) = M(1+t)^{-d}\int_{\mathbb{R}^{N-1}}\left|\langle D\rangle^{[(N-1)/2]+1}Ju(t,x)\right|^2 d\check{x}_j \tag{2.58}
$$

with some constants $M>0$ and $d>0$. On the other hand, in [3], [4] and [5] we chose $\phi_j(t,x_j)$ as

$$
\phi_j(t,x_j) = \psi(x_j) = M\langle x_j\rangle^{-1-\delta} \tag{2.59}
$$

or

$$
\phi_j(t,x_j) = M(1+t)^{-d}\langle x_j\rangle^{-1-\delta} \tag{2.60}
$$

with some constants $M>0$, $\delta>0$ and $d>0$. (2.57) or (2.59) is applied to the local existence. To use (2.59), we need the spatial decay of the solution and then we have to introduce the weighted Sobolev spaces. On the other hand, we make use of (2.58) or (2.60) to obtain the global existence results. (2.60) causes generally loss of time-decay and (2.58)

does not. (2.58) has the structural nice property. In [27] S. Katayama and Y. Tsutsumi made good use of this property to study the global existence theorem for (1.1)–(1.2) in one space dimension ($N=1$). It is more important that $\phi_j(t,\cdot)$ is smooth enough to be treated by the symbolic calculus. Unfortunately, when $F(u,q)$ is quadratic, (2.57) and (2.58) are not applicable to (1.1)–(1.2). It seems to be difficult to find sufficiently smooth functions $\phi_j(t,x_j)$, $j=1,\cdots,N$, which give no loss of time-decay and are applicable to the quadratic nonlinearity.

Chapter 3

Preliminaries

This chapter is devoted to the preliminaries: the estimates of nonlinear term, the time-decay estimates, and the parabolic regularization.

3.1 Gagliardo–Nirenberg–Moser estimates

In this section we obtain the estimates of nonlinear term $F(u, \nabla u)$ in the usual Sobolev spaces H^m ($m \in \mathbb{N}$), in order to prove the local existence theorems. In general, such nonlinear estimates follow from the Sobolev embedding theorems. Throughout of the present chapter, the Gagliardo–Nirenberg inequalities play an important role. These inequalities should be called "the sharp Sobolev embeddings". Historically, L. Nirenberg ([38]) and E. Gagliardo ([15]) first obatained these inequalities and applied to studying nonlinear problems. Now, such application is one of the standard technique in the studies on nonlinear partial differential equations (see e.g., [37, §2] or [31]). By this technique, we can see nonlinear term $f(u)$ as if it were coefficient $\times\, u$ from a viewpoint of estimates. And then, we can treat nonlinear partial diffrential equations by the theory of linear partial diffrential equations.

Now, we introduce the Gagliardo–Nirenberg inequalities.

Lemma 3.1.1 (The Gagliardo–Nirenberg inequality) *Let r_0, r_1 and r_2 satisfy $1 \le r_0, r_1, r_2 \le \infty$. and let j_1 and j_2 be integers satisfying $0 \le j_1 < j_2$. Then there exists a constant $C_0 = C_0(N, j_1, j_2, r_0, r_2, a) > 0$ such that for any $u \in L^{r_0}$ satisfying $\partial^\alpha u \in L^{r_2}$, $|\alpha| = j_2$, the following inequalities hold*

$$\sum_{|\beta|=j_1} \|\partial^\beta u\|_{L^{r_1}} \le C_0 \sum_{|\alpha|=j_2} \|\partial^\alpha u\|_{L^{r_2}}^a \|u\|_{L^{r_0}}^{1-a}, \tag{3.1}$$

where

$$\frac{1}{r_1} = \frac{j_1}{N} + a\left(\frac{1}{r_2} - \frac{j_2}{N}\right) + (1-a)\frac{1}{r_0}$$

for all a in the interval

$$\frac{j_1}{j_2} \le a \le 1.$$

with the following exceptional cases
i) *If $j_1 = 0$, $r_2 j_2 < N$, $r_0 = \infty$, then we make the additional assumption that $u(x) \longrightarrow 0$ as $|x| \to +\infty$.*
ii) *If $1 < r_2 < \infty$, and $j_2 - j_1 - N/r_2 \in \mathbb{Z}_+$, then (3.1) holds only for a satisfying $j_1/j_2 \le a < 1$.*

Proof. See A. Friedman's text book [14, Theorem 9.3] for instance. ∎

Following J. Moser's [37] or S. Klainerman's [31], we obtain the estimates on nonlinear term $F(u, \nabla u)$ to prove the local existence theorem.

Lemma 3.1.2 *We assume $F(u, q) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^{2N} ; \mathbb{C})$. Let m be an integer $> N/2 + 1$. Then there exists a non-decreasing function $A_m(\cdot)$ on $[0, \infty)$ such that for any $u, v \in H^m$,*

$$\begin{aligned}
\|F(u,\nabla u)\|_{H^{m-1}} &\le A_m\Big(\|u\|_{W^{1,\infty}}\Big)\|u\|_{H^m}, && (3.2)\\
\sum_{|\alpha|=m} \|\partial^\alpha F(u,\nabla u) - P(u,\nabla u, \partial\partial^\alpha u)\|_{L^2} &\le A_m\Big(\|u\|_{W^{1,\infty}}\Big)\|u\|_{H^m}, && (3.3)\\
\|F(u,\nabla u) - F(v,\nabla v)\|_{H^{m-1}} &\le A_m\Big(\|u\|_{H^m} + \|v\|_{H^m}\Big)\|u - v\|_{H^m}, && (3.4)
\end{aligned}$$

where

$$P(u, \nabla u, \partial\partial^\alpha u) = \sum_{j=1}^N \frac{\partial F}{\partial q_j}(u,\nabla u)\partial_j \partial^\alpha u + \sum_{j=1}^N \frac{\partial F}{\partial \bar{q}_j}(u,\nabla u)\partial_j\partial^\alpha \bar{u}.$$

Proof. Making strong use of the Gagliardo–Nirenberg inequalities, we prove Lemma 3.1.2. Since $C_0^\infty(\mathbb{R}^N)$ is densely embedded in $H^m(\mathbb{R}^N)$, the following calculations are valid via the smooth approximation.

First we show (3.2). We note here that

$$\|F(u,\nabla u)\|_{H^{m-1}} \le C\Big(\|F(u,\nabla u)\|_{L^2} + \sum_{|\alpha|=m-1} \|\partial^\alpha F(u,\nabla u)\|_{L^2}\Big). \tag{3.5}$$

Because $F(0,0) = 0$, the mean value theorem gives

$$\begin{aligned}
F(u,\nabla u) &= \left(\int_0^1 \frac{\partial F}{\partial u}(\theta u, \theta\nabla u)d\theta\right) u\\
&+ \left(\int_0^1 \frac{\partial F}{\partial \bar{u}}(\theta u, \theta\nabla u)d\theta\right) \bar{u}\\
&+ \sum_{j=1}^N \left(\int_0^1 \frac{\partial F}{\partial q_j}(\theta u, \theta\nabla u)d\theta\right)\partial_j u\\
&+ \sum_{j=1}^N \left(\int_0^1 \frac{\partial F}{\partial \bar{q}_j}(\theta u, \theta\nabla u)d\theta\right)\partial_j \bar{u}.
\end{aligned}$$

Then, there exists a non-decreasing function $A(\cdot)$ on $[0, +\infty)$ such that

$$\begin{aligned}
\|F(u,\nabla u)\|_{L^2} \le& \left\{\sup_{\theta\in[0,1]}\left\|\frac{\partial F}{\partial u}(\theta u, \theta\nabla u)\right\|_{L^\infty} + \sup_{\theta\in[0,1]}\left\|\frac{\partial F}{\partial \bar{u}}(\theta u, \theta\nabla u)\right\|_{L^\infty}\right\}\|u\|_{L^2}\\
&+ \sum_{j=1}^N \left\{\sup_{\theta\in[0,1]}\left\|\frac{\partial F}{\partial q_j}(\theta u, \theta\nabla u)\right\|_{L^\infty} + \sup_{\theta\in[0,1]}\left\|\frac{\partial F}{\partial \bar{q}_j}(\theta u, \theta\nabla u)\right\|_{L^\infty}\right\}\|\nabla u\|_{L^2}\\
\le& A\Big(\|u\|_{W^{1,\infty}}\Big)\|u\|_{H^1}.
\end{aligned} \tag{3.6}$$

Let α be a multi-index with $|\alpha| = m - 1$. Using the Leibniz' formula, we have

$$\begin{aligned}
\partial^\alpha F(u,\nabla u) &= \sum_* \frac{\partial^{\beta_0+\bar\beta_0+\beta+\bar\beta}F}{\partial u^{\beta_0}\partial\bar{u}^{\bar\beta_0}\partial q^\beta \partial\bar{q}^{\bar\beta}}(u,\nabla u)\\
&\times \prod_{\alpha_0\le\alpha} (\partial^{\alpha_0}u)^{m(\alpha_0)}\\
&\times \prod_{j=1}^N \prod_{\alpha_j\le\alpha} (\partial^{\alpha_0}\partial_j u)^{m(\alpha_j)}\\
&\times \prod_{\bar\alpha_0\le\alpha} (\partial^{\bar\alpha_0}\bar{u})^{m(\alpha_0)}\\
&\times \prod_{j=1}^N \prod_{\bar\alpha_j\le\alpha} (\partial^{\alpha_j}\partial_j \bar{u})^{m(\alpha_j)},
\end{aligned}$$

where $\beta_0, \bar\beta_0, m(\alpha_n), m(\bar\alpha_n) \in \mathbb{Z}_+ (n = 0, 1, \cdots, N)$, $\beta, \bar\beta, \alpha_n, \bar\alpha_n \in (\mathbb{Z}_+)^N (n = 0, 1, \cdots, N)$, and $\sum_*$ means summing up on $1 \le \beta_0 + \bar\beta_0 + |\beta| + |\bar\beta| \le |\alpha|$ restricted to

$$\begin{aligned}
&\sum_{\alpha_0\le\alpha} m(\alpha_0) = \beta_0, \qquad \sum_{j=1}^N \sum_{\alpha_j\le\alpha} m(\alpha_j) = |\beta|,\\
&\sum_{\alpha_0\le\alpha} m(\bar\alpha_0) = \bar\beta_0, \qquad \sum_{j=1}^N \sum_{\alpha_j\le\alpha} m(\bar\alpha_j) = |\bar\beta|,\\
&\sum_{\alpha_0\le\alpha} m(\alpha_0)\alpha_0 + \sum_{j=1}^N \sum_{\alpha_j\le\alpha} m(\alpha_j)\alpha_j\\
+ &\sum_{\bar\alpha_0\le\alpha} m(\bar\alpha_0)\bar\alpha_0 + \sum_{j=1}^N \sum_{\alpha_j\le\alpha} m(\bar\alpha_j)\alpha_j = \alpha.
\end{aligned}$$

By the Hörder inequality, we have

$$\begin{aligned}
\|\partial^\alpha F(u,\nabla u)\|_{L^2} &\le A\Big(\|u\|_{W^{1\infty}}\Big)\sum_{*} \\
&\times \prod_{\alpha_0\le\alpha} \|\partial^{\alpha_0} u\|^{m(\alpha_0)}_{L^{2(m-1)/|\alpha_0|}} \\
&\times \prod_{\bar\alpha_0\le\alpha} \|\partial^{\alpha_0} u\|^{m(\alpha_0)}_{L^{2(m-1)/|\bar\alpha_0|}} \\
&\times \prod_{j=1}^{N}\prod_{\alpha_j\le\alpha} \|\partial^{\alpha_j}\partial_j u\|^{m(\alpha_j)}_{L^{2(m-1)/|\alpha_j|}} \\
&\times \prod_{j=1}^{N}\prod_{\alpha_0\le\alpha} \|\partial^{\bar\alpha_j}\partial_j \bar u\|^{m(\bar\alpha_j)}_{L^{2(m-1)/|\bar\alpha_j|}}.
\end{aligned}$$

The Gagliardo–Nirenberg inequalities imply

$$\begin{aligned}
\|\partial^\alpha F(u,\nabla u)\|_{L^2} &\le A\Big(\|u\|_{W^{1\infty}}\Big)\sum_{*} \\
&\times \prod_{\alpha_0\le\alpha} \|u\|^{m(\alpha_0)(1-|\alpha_0|/(m-1))}_{L^\infty}\||D|u\|^{m(\alpha_0|\alpha_0|/(m-1))}_{L^2} \\
&\times \prod_{\alpha_0\le\alpha} \|\bar u\|^{m(\bar\alpha_0)(1-|\bar\alpha_0|/(m-1))}_{L^\infty}\||D|\bar u\|^{m(\bar\alpha_0|\alpha_0|/(m-1))}_{L^2} \\
&\times \prod_{j=1}^{N}\prod_{\alpha_j\le\alpha} \|\partial_j u\|^{m(\alpha_j)(1-|\alpha_j|/(m-1))}_{L^\infty}\||D|\partial_j u\|^{m(\alpha_j|\alpha_j|/(m-1))}_{L^2} \\
&\times \prod_{j=1}^{N}\prod_{\alpha_j\le\alpha} \|\partial_j \bar u\|^{m(\alpha_j)(1-|\alpha_j|/(m-1))}_{L^\infty}\||D|\partial_j \bar u\|^{m(\bar\alpha_j|\alpha_j|/(m-1))}_{L^2} \\
&\le A\Big(\|u\|_{W^{1\infty}}\Big)\sum_{*} \\
&\times \|u\|_{W^{1,\infty}}^{\sum(m(\alpha_n)+m(\bar\alpha_n)-\sum(m(\alpha_n)|\alpha_n|+m(\bar\alpha_n)|\bar\alpha_n|)/(m-1)} \\
&\times \|u\|_{H^m}^{\sum(m(\alpha_n)|\alpha_n|+m(\bar\alpha_n)|\bar\alpha_n|)/(m-1)} \\
&= A\Big(\|u\|_{W^{1\infty}}\Big)\sum_{*}\|u\|_{W^{1,\infty}}^{\beta_0+\bar\beta_0+|\beta|+|\bar\beta|-1}\|u\|_{H^m} \\
&\le A_m\Big(\|u\|_{W^{1\infty}}\Big)\|u\|_{H^m}.
\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we obtain (3.2).

Secondly we give the outline of the proof of (3.3) and (3.4). The Leibniz formula and the mean value theorem imply

$$\begin{aligned}
&\partial^\alpha F(u,\nabla u) - P(u,\nabla u,\partial\partial^\alpha) \\
&= \sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\partial^\beta\frac{\partial F}{\partial u}(u,\nabla u)\partial^{\alpha-\beta}u \\
&+ \sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\partial^\beta\frac{\partial F}{\partial \bar u}(u,\nabla u)\partial^{\alpha-\beta}\bar u \\
&+ \sum_{j=1}^{N}\sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\partial^\beta\frac{\partial F}{\partial q_j}(u,\nabla u)\partial^{\alpha-\beta}\partial_j u \\
&+ \sum_{j=1}^{N}\sum_{\beta\le\alpha'}\binom{\alpha'}{\beta}\partial^\beta\frac{\partial F}{\partial \bar q_j}(u,\nabla u)\partial^{\alpha-\beta}\partial_j \bar u, \\
&\partial^\alpha\Big(F(u,\nabla u)-F(v,\nabla v)\Big) \\
&= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\left(\int_0^1\frac{\partial F}{\partial u}(\theta u+(1-\theta)v,\theta\nabla u+(1-\theta)\nabla v)d\theta\right)\partial^{\alpha-\beta}(u-v) \\
&+ \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\left(\int_0^1\frac{\partial F}{\partial \bar u}(\theta u+(1-\theta)v,\theta\nabla u+(1-\theta)\nabla v)d\theta\right)\partial^{\alpha-\beta}(\bar u-\bar v) \\
&+ \sum_{j=1}^{N}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\left(\int_0^1\frac{\partial F}{\partial q_j}(\theta u+(1-\theta)v,\theta\nabla u+(1-\theta)\nabla v)d\theta\right)\partial^{\alpha-\beta}\partial_j(u-v) \\
&+ \sum_{j=1}^{N}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\left(\int_0^1\frac{\partial F}{\partial \bar q_j}(\theta u+(1-\theta)v,\theta\nabla u+(1-\theta)\nabla v)d\theta\right)\partial^{\alpha-\beta}\partial_j(\bar u-\bar v),
\end{aligned}$$

where $\alpha'\in(\mathbb{Z}_+)^N$ satisfies $|\alpha-\alpha'|=1$. Using the above formulae, we can obtain (3.3) and (3.4) in the same way as (3.2). ∎

3.2 Decay estimates for cubic semilinear equations

The present section is concerned with the decay estimates of solutions to cubic semilinear Schrödinger equations. Thourghout the present section, We assume the cubic nonlinearity:

$$F(u,q) = O\Big(|u|^3+|q|^3\Big) \quad \text{near} \quad (u,q)=0.$$

Combining the local existence theorem and the decay estimates, we prove the global existence theorem. The later come from the decay estimates of the fundamental solution $e^{it\Delta}$ essentially. In fact, we can express the operator $e^{it\Delta}$ by the explicit formula

$$\Big(e^{it\Delta}\phi\Big)(x) = (4\pi it)^{-N/2}\int_{\mathbb{R}^N}e^{-|x-y|^2/4it}\phi(y)dy \quad \text{for} \quad \phi\in S(\mathbb{R}^N). \tag{3.8}$$

Immediatly we get

$$\|e^{it\Delta}\phi\|_{L^\infty}\le C|t|^{-N/2}\|\phi\|_{L^1} \quad \text{for} \quad t\in\mathbb{R}\backslash\{0\}. \tag{3.9}$$

In the next section we make strong use of (3.9) to get the decay estimates of solutions to quadratic semilinear Schrödinger equations. On the other hand, we employ another method which consists of the Gagliardo–Nirenberg inequalities and the operator $J = (J_1, \cdots, J_N)$ defined by

$$J_k u = (x_k + 2i(1+t)\partial_k)u = e^{ix_k^2/4(1+t)} 2i\partial_k\left(e^{-ix_k^2/4(1+t)}u\right)$$

for $u \in \mathcal{S}(\mathbb{R}^N)$. Such decay esimates are well-known in the studies on the global existence of small solutions to semilienar Schrödinger equations. This method suits with our energy estimates when we treat cubic semilinear Schrödinger equations. We note here the important commutation relations:

$$[J_k, \partial_t - i\Delta] = 0, \quad [J_k, J_l] = 0, \quad [\partial_k, J_l] = \delta_{kl} \quad (k, l = 1, \cdots, N),$$

where δ_{kl} is Kronecker's delta.

To get the decay estimates we prepare

Lemma 3.2.1 (1) *Let $N = 1$. For any $u \in H^1 \cap H^{0,1}$,*

$$\|u\|_{L^\infty} \le C(1+t)^{-1/2}\|J_1 u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}. \tag{3.10}$$

(2) *Let $N = 2$. For any $v \in H^2 \cap H^{1,1}$,*

$$\|v\|_{L^\infty} \le C(1+t)^{-3/4} \sum_{\substack{|\alpha|\le 1\\ |\beta|=1}} \|\partial^\alpha J^\beta v\|_{L^2}^{3/4}\|v\|_{L^2}^{1/4}. \tag{3.11}$$

(3) *Let $N = 3$. For any $w \in H^3 \cap H^{1,2}$,*

$$\|w\|_{L^\infty} \le C(1+t)^{-3/2} \sum_{|\beta|=2} \|J^\beta w\|_{L^2}^{1/2} \sum_{|\beta'|=1} \|J^{\beta'} w\|_{L^2}^{1/2}, \tag{3.12}$$

$$\sum_{|\beta'|=1} \|J^{\beta'} w\|_{L^\infty} \le C(1+t)^{-1} \sum_{\substack{|\alpha|\le 1\\ |\beta|=2}} \|\partial^\alpha J^\beta w\|_{L^2}. \tag{3.13}$$

Proof. Let $\theta = \theta(t,x) = \theta(t,x_1) + \cdots + \theta(t,x_N) = |x|^2/4(1+t)$.

To show (3.10), we use (3.1) with $N = 1$, $r_0 = 2$, $r_1 = \infty$, $r_2 = 2$, $j_1 = 0$, $j_2 = 1$ and $a = 1/2$. Then we have

$$\begin{aligned}
\|u\|_{L^\infty} &= \|e^{-i\theta}u\|_{L^\infty} \\
&\le C\|\partial_1(e^{-i\theta}u)\|_{L^2}^{1/2}\|e^{-i\theta}u\|_{L^2}^{1/2} \\
&= C\|e^{i\theta}\partial_1(e^{-i\theta}u)\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2} \\
&\le C(1+t)^{-1/2}\|J_1 u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}.
\end{aligned}$$

Next we show (3.11). By (3.1) with $N = 2$, $r_0 = 4$, $r_1 = \infty$, $r_2 = 4$, $j_1 = 0$, $j_2 = 1$ and $a = 1/2$, we get

$$\begin{aligned}
\|v\|_{L^\infty} &= \|e^{-i\theta}v\|_{L^\infty} \\
&\le C(1+t)^{-1/2} \sum_{|\beta|=1} \|J^\beta v\|_{L^4}^{1/2}\|v\|_{L^4}^{1/2}.
\end{aligned} \tag{3.14}$$

Making use of (3.1) again with $N = 2$, $r_0 = 2$, $r_1 = 4$, $r_2 = 2$, $j_1 = 0$, $j_2 = 1$ and $a = 1/2$, we have

$$\sum_{|\beta|=1} \|J^\beta v\|_{L^4} \le \sum_{\substack{|\alpha|=1\\ |\beta|=1}} \|\partial^\alpha J^\beta v\|_{L^2}^{1/2} \sum_{|\beta'|=1} \|J^{\beta'} v\|_{L^2}^{1/2}, \tag{3.15}$$

$$\|v\|_{L^4} \le C(1+t)^{-1/2} \sum_{|\beta|=1} \|J^\beta v\|_{L^2}^{1/2}\|v\|_{L^2}^{1/2}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain (3.11).

For (3.12), by (3.1) with $N = 3$, $r_0 = 6$, $r_1 = \infty$, $r_2 = 6$, $j_1 = 0$, $j_2 = 1$ and $a = 1/2$, we have

$$\|w\|_{L^\infty} \le C(1+t)^{-1/2} \sum_{|\beta|=1} \|J^\beta w\|_{L^6}^{1/2}\|v\|_{L^6}^{1/2}.$$

Making use of (3.1) with $N = 3$, $r_0 = 2$, $r_1 = 6$, $r_2 = 2$, $j_1 = 0$, $j_2 = 1$ and $a = 1$ once again, we can obtain (3.12). Similarly we can show (3.13). ∎

Let $F_3(u,q)$ be a homogeneous cubic part of $F(u,q)$ near $(u,q) = 0$. We put $F_4(u,q) = F(u,q) - F_3(u,q)$. When the spatial dimension N is equal to 2, we make use of this decomposition. For the sake of convenience, we introduce the following notations

$$X_m^N(t) = \begin{cases} \displaystyle\sum_{\substack{|\alpha+2\beta|\le m\\ |\beta|\le 1}} \|\partial^\alpha J^\beta u(t)\|_{L^2} + (1+t)^{-1/4} \sum_{\substack{|\alpha+2\beta|\le m\\ |\beta|=2}} \|\partial^\alpha J^\beta u(t)\|_{L^2}, & \text{if } N \ge 3, \\ \displaystyle\sum_{|\alpha|\le m} \|\partial^\alpha u(t)\|_{L^2} + \sum_{\substack{|\alpha|\le 3\\ |\beta|=1}} \|\partial^\alpha J^\beta u(t)\|_{L^2}, & \text{if } N = 2, \end{cases}$$

$$\text{where} \qquad m \ge \begin{cases} m_0 = \left[\dfrac{N+1}{2}\right] + 6, & \text{if } N \ge 3, \\ 7, & \text{if } N = 2, \end{cases}$$

$$\begin{aligned}
P_\alpha &= P(u, \nabla u, \partial\partial^\alpha u) \\
&= \sum_{j=1}^N \frac{\partial F}{\partial q_j}(u, \nabla u)\partial_j\partial^\alpha u + \sum_{j=1}^N \frac{\partial F}{\partial \bar{q}_j}(u, \nabla u)\partial_j\partial^\alpha \bar{u}, \\
P_{\alpha\beta} &= P'(u, \nabla u, \partial\partial^\alpha J^\beta u)
\end{aligned}$$

$$\begin{cases} \sum_{j=1}^{N} G_j(u,\nabla u)\partial_j\partial^\alpha J^\beta u + \sum_{j=1}^{N} G_j'(u,\nabla u)\partial_j\overline{\partial^\alpha J^\beta u}, & \text{if} \quad N \geq 3, \\ \sum_{j=1}^{N} \frac{\partial F_3}{\partial q_j}(u,\nabla u)\partial_j\partial^\alpha J^\beta u + (-1)^{|\beta|}\sum_{j=1}^{N} \frac{\partial F_3}{\partial \bar{q}_j}(u,\nabla u)\partial_j\overline{\partial^\alpha J^\beta u} & \\ +\sum_{j=1}^{N} G_{4,j}(u,\nabla u)\partial_j\partial^\alpha J^\beta u + \sum_{j=1}^{N} G_{4,j}'(u,\nabla u)\partial_j\overline{\partial^\alpha J^\beta u}, & \text{if} \quad N = 2, \end{cases}$$

$$\begin{aligned} G_0(u,\nabla u) &= \int_0^1 \frac{\partial F}{\partial u}(\theta u,\theta\nabla u)d\theta, & G_0'(u,\nabla u) &= \int_0^1 \frac{\partial F}{\partial \bar{u}}(\theta u,\theta\nabla u)d\theta, \\ G_j(u,\nabla u) &= \int_0^1 \frac{\partial F}{\partial q_j}(\theta u,\theta\nabla u)d\theta, & G_j'(u,\nabla u) &= \int_0^1 \frac{\partial F}{\partial \bar{q}_j}(\theta u,\theta\nabla u)d\theta, \\ G_{4,j}(u,\nabla u) &= \int_0^1 \frac{\partial F_4}{\partial q_j}(\theta u,\theta\nabla u)d\theta, & G_{4,j}'(u,\nabla u) &= \int_0^1 \frac{\partial F}{\partial \bar{q}_j}(\theta u,\theta\nabla u)d\theta \end{aligned}$$

for $j = 1,\cdots,N$. Using Lemma 3.2.1 and the Sobolev embedding $H^{[n/2]+1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{N})$, we have

$$\begin{aligned} \|u(t)\|_{W^{2,\infty}} &\leq C(1+t)^{-3/4}X_7^2(t), & (3.17) \\ \|Ju(t)\|_{W^{1,\infty}} &\leq CX_7^2(t), & (3.18) \end{aligned}$$

$$\text{for} \quad N=2, \quad t\in[0,T] \quad \text{and} \quad u \in C([0,T];H^7)\cap C([0,T];H^{3,1}),$$

and

$$\begin{aligned} \|u(t)\|_{W^{3,\infty}} &\leq C(1+t)^{-11/8}X_{m_0}^N(t), & (3.19) \\ \|Ju(t)\|_{W^{2,\infty}} &\leq C(1+t)^{-1/2}X_{m_0}^N(t), & (3.20) \\ \|J^2u(t)\|_{W^{1,\infty}} &\leq C(1+t)^{1/4}X_{m_0}^N(t), & (3.21) \end{aligned}$$

$$\text{for} \quad N\geq 3, \quad t\in[0,T] \quad \text{and} \quad u\in\bigcap_{j=0}^{2} C([0,T];H^{m_0-2j,j}).$$

We use the abridged notations of the nonlinear terms, i.e., we abbreviate $F(u(t),\nabla u(t))$ as $F(t)$ for instance. The properties of the nonlinear term $F(u,\nabla u)$ are the following.

Lemma 3.2.2 (1) *Let $N\geq 3$, $m\geq m_0$, $p\geq 3$ and R be an arbitrary positive constant. Then there exists a constant $C = C(R) > 0$ such that*

$$\begin{aligned} \|F(t)\|_{H^{m-1}} + \sum_{|\alpha|=m} \|(\partial^\alpha F - P_\alpha)(t)\|_{L^2} &\leq C(1+t)^{-11/4}X_m^N(t)^3, & (3.22) \\ \|JF(t)\|_{H^{m-3}} + \sum_{\substack{|\alpha|=m-2 \\ |\beta|=1}} \|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} &\leq C(1+t)^{-7/4}X_m^N(t)^3, & (3.23) \\ \|J^2F(t)\|_{H^{m-5}} + \sum_{\substack{|\alpha|=m-4 \\ |\beta|=2}} \|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} &\leq C(1+t)^{-3/4}X_m^N(t)^3, & (3.24) \end{aligned}$$

$$\textit{for} \quad u\in\bigcap_{j=0}^{2} C([0,T];H^{m-2j,j}) \quad \textit{satisfying} \quad \sup_{t\in[0,T]} X_m^N(t)\leq R.$$

(2) *Let $N=2$, $m\geq 7$, $p\geq 3$, $F_3(u,q)$ satisfy the gauge invariance*

$$F_3(e^{i\theta}u, e^{i\theta}q) = e^{i\theta}F(u,q) \quad \textit{for} \quad (u,q)\in\mathbb{C}\times\mathbb{C}^N,\ \theta\in\mathbb{R}, \tag{3.25}$$

and R be an arbitrary positive constant. Then there exists a constant $C = C(R) > 0$ such that

$$\begin{aligned} \|F(t)\|_{H^{m-1}} + \sum_{|\alpha|=m} \|(\partial^\alpha F - P_\alpha)(t)\|_{L^2} &\leq C(1+t)^{-3/2}X_m^2(t)^3, & (3.26) \\ \|JF(t)\|_{H^2} + \sum_{\substack{|\alpha|=3 \\ |\beta|=1}} \|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} &\leq C(1+t)^{-21/20}X_m^2(t)^3, & (3.27) \end{aligned}$$

$$\textit{for} \quad u\in C([0,T];H^m)\cap C([0,T];H^{3,1}) \quad \textit{satisfying} \quad \sup_{t\in[0,T]} X_m^2(t)\leq R.$$

Proof. Let us consider the part (1). In the same way as in Lemma 3.1.2 we have

$$\|F(t)\|_{H^{m-1}} + \sum_{|\alpha|=m} \|(\partial^\alpha F - P_\alpha)(t)\|_{L^2} \leq C\|u(t)\|_{W^{1,\infty}}^2 \|u(t)\|^{H^m}.$$

Then (3.19) yields (3.22).

To show (3.23) and (3.24), we need the accurate forms of JF and J^2F such as

$$\begin{aligned} & J_kF(u,\nabla u) \\ = {} & \sum_{j=1}^{N} G_j(u,\nabla u)\partial_j J_k u + \sum_{j=1}^{N} G_j'(u,\nabla u)\partial_j\overline{J_k u} \\ + {} & F_0^k(u,\nabla u, Ju) + (1+t)F_1^k(u,\nabla u,\nabla^2 u), \\ & J_lJ_kF(u,\nabla u) \\ = {} & \sum_{j=1}^{N} G_j(u,\nabla u)\partial_j J_lJ_k u + \sum_{j=1}^{N} G_j'(u,\nabla u)\partial_j\overline{J_lJ_k u} \\ + {} & F_0^{lk}(u,\nabla u, Ju, J^2u) + (1+t)F_1^{lk}(u,\nabla u,\nabla^2 u, Ju, \nabla Ju, \nabla^2 Ju) \\ + {} & (1+t)^2F_2^{lk}(u,\nabla u,\nabla^2 u,\nabla^3 u), \end{aligned}$$

where

$$\begin{aligned} & F_0^k(u,\nabla u, Ju) \\ = {} & G_0(u,\nabla u)J_ku + G_0'(u,\nabla u)\overline{J_k u} \end{aligned}$$

$$\begin{aligned}
- &\quad G_k(u,\nabla u)u - G'_k(u,\nabla u)\bar{u}.\\
&\quad F_1^k(u,\nabla u,\nabla^2 u)\\
= &\quad 2i\Big\{\partial_k G_0(u,\nabla u)u + \partial_k G'_0(u,\nabla u)\bar{u}\\
&\qquad + \sum_{j=1}^{N}\partial_k G_j(u,\nabla u)\partial_j u + \sum_{j=1}^{N}\partial_k G'_j(u,\nabla u)\partial_j \bar{u}\Big\}\\
&\qquad + 4i\Big\{G'_0(u,\nabla u)\partial_k\bar{u} + \sum_{j=1}^{N} G'_j(u,\nabla u)\partial_j\partial_k\bar{u}\Big\},
\end{aligned}$$

$$\begin{aligned}
&F_0^{lk}(u,\nabla u, Ju, J^2u)\\
= &\quad G_0(u,\nabla u)J_lJ_ku + G'_0(u,\nabla u)\overline{J_lJ_ku}\\
- &\quad G_k(u,\nabla u)J_lu - G'_k(u,\nabla u)\overline{J_lu}\\
- &\quad G_l(u,\nabla u)J_ku - G'_l(u,\nabla u)\overline{J_ku},
\end{aligned}$$

$$\begin{aligned}
&F_1^{lk}(u,\nabla u,\nabla^2u, Ju, \nabla Ju, \nabla^2 Ju)\\
= &\quad 2i\Big\{\partial_l G_0(u,\nabla u)J_ku + \partial_l G'_0(u,\nabla u)\overline{J_ku}\\
&\qquad + \sum_{j=1}^{N}\partial_l G_j(u,\nabla u)\partial_j J_ku + \sum_{j=1}^{N}\partial_l G'_j(u,\nabla u)\partial_j\overline{J_ku}\\
&\qquad + \partial_k G_0(u,\nabla u)J_lu + \partial_k G'_0(u,\nabla u)\overline{J_lu}\\
&\qquad + \sum_{j=1}^{N}\partial_k G_j(u,\nabla u)\partial_j J_lu + \sum_{j=1}^{N}\partial_k G'_j(u,\nabla u)\partial_j\overline{J_lu}\\
&\qquad - \partial_l G_k(u,\nabla u)u - \partial_l G'_k(u,\nabla u)\bar{u}\\
&\qquad - \partial_k G_l(u,\nabla u)u - \partial_k G'_l(u,\nabla u)\bar{u}\Big\}\\
+ &\quad 4i\Big\{G'_0(u,\nabla u)\partial_l\overline{J_ku} + \sum_{j=1}^{N} G'_j(u,\nabla u)\partial_j\partial_l\overline{J_ku}\\
&\qquad + G'_0(u,\nabla u)\partial_k\overline{J_lu} + \sum_{j=1}^{N} G'_j(u,\nabla u)\partial_j\partial_k\overline{J_lu}\\
&\qquad - G'_k(u,\nabla u)\partial_l\bar{u} - G'_l(u,\nabla u)\partial_k\bar{u}\\
&\qquad - G'_0(u,\nabla u)\bar{u}\delta_{lk} - \sum_{j=1}^{N} G'_j(u,\nabla u)\partial_j\bar{u}\delta_{lk}\Big\},
\end{aligned}$$

$$\begin{aligned}
&F_2^{lk}(u,\nabla u,\nabla^2u,\nabla^3u)\\
= &\quad -4\Big\{\partial_l\partial_k G_0(u,\nabla u)u + \partial_l\partial_k G'_0(u,\nabla u)\bar{u}\\
&\qquad + \sum_{j=1}^{N}\partial_l\partial_k G_j(u,\nabla u)\partial_j u + \sum_{j=1}^{N}\partial_l\partial_k G'_j(u,\nabla u)\partial_j u\Big\}\\
- &\quad 8\Big\{\partial_k G'_0(u,\nabla u)\partial_l\bar{u} + \sum_{j=1}^{N}\partial_k G'_j(u,\nabla u)\partial_j\partial_l\bar{u}\\
&\qquad + \partial_l G'_0(u,\nabla u)\partial_k\bar{u} + \sum_{j=1}^{N}\partial_l G'_j(u,\nabla u)\partial_j\partial_k u\Big\}\\
- &\quad 16\Big\{G'_0(u,\nabla u)\partial_l\partial_k\bar{u} + \sum_{j=1}^{N} G'_j(u,\nabla u)\partial_j\partial_l\partial_k u\Big\},
\end{aligned}$$

for $k,l = 1,\cdots,N$.

We have only to see JF as

$$\begin{aligned}
JF &= Q_0(u,\nabla u, Ju, \nabla Ju) + (1+t)C_0(u,\nabla u,\nabla^2u),\\
Q_0 &= \big(\text{quadratic term of } (u,\nabla u)\big)\times(u,\nabla u, Ju,\nabla Ju),\\
C_0 &= \text{cubic term of } (u,\nabla u,\nabla^2 u).
\end{aligned}$$

In the same manner as in Lemma 3.1.2, we have

$$\begin{aligned}
\|Q_0(t)\|_{H^{m-3}} &\le C\sum_{n=0}^{m-3}\|u(t)\|_{W^{1,\infty}}^{2-\frac{n}{m-3}}\|u(t)\|_{H^{m-2}}^{\frac{n}{m-3}}\|Ju(t)\|_{W^{1,\infty}}^{\frac{n}{m-3}}\|Ju(t)\|_{H^{m-2}}^{1-\frac{n}{m-3}}\\
&\le C(1+t)^{-15/8}X_m^N(t)^3,\\
\|(1+t)C_0(t)\|_{H^{m-2}} &\le C(1+t)^{-7/4}X_m^N(t)^3.
\end{aligned}$$

Then we get

$$\|JF(t)\|_{H^{m-3}} \le C(1+t)^{-7/4}X_m^N(t)^3. \tag{3.28}$$

Let α and $\beta\in(\mathbb{Z}_+)^N$ satisfy $|\alpha| = m-2$ and $|\beta| = 1$ respectively. The simple calculation gives

$$\partial^\alpha J^\beta F - P_{\alpha\beta} = (\partial^\alpha P_{0\beta} - P_{\alpha\beta}) + \sum_{|\alpha'| = m-2}\Big(\partial^{\alpha'}Q_{0\alpha'} + (1+t)\partial^{\alpha'}C_{0\alpha'}\Big).$$

We remark that the structures of $Q_{0\alpha'}$ and $C_{0\alpha'}$ are the same as those of Q_0 and C_0 respectively, and furthermore $Q_{0\alpha'}$ does not contain ∇Ju. Then we obtain

$$\sum_{\substack{|\alpha| = m-2\\|\beta| = 1}}\|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} \le C(1+t)^{-7/4}X_m^N(t)^3. \tag{3.29}$$

(3.28)–(3.29) shows (3.23).

Next we obtain (3.24). We can see J^2F as

$$J^2F = Q_1(u,\nabla u, Ju, J^2u, \nabla J^2u)$$

$$
\begin{aligned}
&+ (1+t)Q_2(u, \nabla u, \nabla^2 u, Ju, \nabla Ju, \nabla^2 Ju) \\
&+ (1+t)^2 C_1(u, \nabla u, \nabla^2 u, \nabla^3 u), \\
Q_1 &\quad \big(\text{quadratic term of } (u, \nabla u)\big) \times (Ju, J^2 u, \nabla J^2 u), \\
Q_2 &\quad \big(\text{quadratic term of } (u, \nabla u, \nabla^2 u)\big) \times (u, \nabla u, Ju, \nabla Ju, \nabla^2 Ju), \\
C_1 &\quad \text{cubic term of } (u, \nabla u, \nabla^2 u, \nabla^3 u).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\|Q_1(t)\|_{H^{m-5}} &\le C \sum_{n=0}^{m-5} \|u(t)\|_{W^{1,\infty}}^{2-\frac{n}{m-5}} \|u(t)\|_{H^{m-4}}^{\frac{n}{m-5}} \\
&\quad \times \sum_{k=1,2} \|J^k u(t)\|_{W^{1,\infty}}^{\frac{n}{m-5}} \sum_{k'=1,2} \|J^{k'} u(t)\|_{H^{m-4}}^{1-\frac{n}{m-5}} \\
&\le C(1+t)^{-9/8} X_m^N(t)^3, \\
\|(1+t)Q_2(t)\|_{H^{m-4}} &\le C(1+t)^{-7/8} X_m^N(t)^3, \\
\|(1+t)^2 C_1(t)\|_{H^{m-4}} &\le C(1+t)^{-3/4} X_m^N(t)^3.
\end{aligned}
$$

Then we get

$$
\|J^2 F(t)\|_{H^{m-5}} \le C(1+t)^{-3/4} X_m^N(t)^3. \tag{3.30}
$$

For any α and $\beta \in (\mathbb{Z}_+)^N$ satisfy $|\alpha| = m-2$ and $|\beta| = 1$, we have

$$
\begin{aligned}
\partial^\alpha J^\beta F - P_{\alpha\beta} &= (\partial^\alpha P_{0\beta} - P_{\alpha\beta}) \\
&+ \sum_{|\alpha'| = m-4} \Big(\partial^{\alpha'} Q_{1\alpha'} + (1+t)\partial^{\alpha'} C_{2\alpha'} + (1+t)^2 \partial^{\alpha'} C_{1\alpha'}\Big).
\end{aligned}
$$

The structures of $Q_{1\alpha'}$, $Q_{2\alpha'}$ and $C_{1\alpha'}$ are the same as those of Q_1, Q_2 and C_1 respectively, and $Q_{1\alpha'}$ does not contain $\nabla J^2 u$. Then we obtain

$$
\sum_{\substack{|\alpha| = m-4 \\ |\beta| = 2}} \|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} \le C(1+t)^{-3/4} X_m^N(t)^3. \tag{3.31}
$$

(3.30) (3.31) shows (3.24).

Now we consider the part (2). Let the spatial dimension N be equal to 2. We make use of the estimates (3.16) and (3.18). In the same way as (3.22), we can show (3.26). But there are some differences between the proofs of (3.23) and of (3.27). First we remark that because $F_3(u, q)$ satisfies (3.25), the loss of decay does not take place in $F_3(u, \nabla u)$, because J acts on $F_3(u, \nabla u)$ as if it were a usual differentiation ∂ in a sense. More precisely, the formula like the Leibniz' one holds

$$
\begin{aligned}
JF_3(u, \nabla u) &= \frac{\partial F_3}{\partial u}(u, \nabla u) Ju - \frac{\partial F_3}{\partial \bar{u}}(u, \nabla u) \overline{Ju} \\
&+ \sum_{j=1}^{N} \frac{\partial F_3}{\partial q_j}(u, \nabla u) J\partial_j u - \sum_{j=1}^{N} \frac{\partial F_3}{\partial \bar{q}_j}(u, \nabla u) \overline{J\partial_j u}.
\end{aligned}
$$

Then the simple calculation gives

$$
\begin{aligned}
JF &= Q_3(u, \nabla u, Ju, \nabla Ju) + (1+t)C_3(u, \nabla u, \nabla^2 u), \\
Q_3 &= Q_4(u, \nabla u) \times (u, \nabla u, Ju, \nabla Ju), \\
Q_4 &= \text{quadratic term of } (u, \nabla u), \\
C_3 &= \text{the forth order term of } (u, \nabla u, \nabla^2 u).
\end{aligned}
$$

In the same way as the evaluation to $C_0(t)$ we can estimate $C_3(t)$. On the other hand, if we try to get the bound of $Q_3(t)$ by the same technique used for $Q_0(t)$, then the lack of decay occurs. To avoid this difficulty we make use of another estimates. We have

$$
\|Q_3(t)\|_{H^2} \le C\|Q_4(t)\|_{W^{2,\infty}} \Big(\|u(t)\|_{H^2} + \|Ju(t)\|_{H^3}\Big). \tag{3.32}
$$

Similarly when we try to obtain

$$
\sum_{\substack{|\alpha|=3 \\ |\beta|=1}} \|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} \le C(1+t)^{-21/20} X_m^2(t)^3, \tag{3.33}
$$

we need the bound which is like $VertQ_4(t)\|_{W^{3,\infty}}$. In the same way as (3.2), we get

$$
\|Q_4(t)\|_{W^{3,\infty}} \le C \sum_{n=0}^{3} \|u(t)\|_{H^7}^{\frac{n}{5}} \|u(t)\|_{W^{1,\infty}}^{2-\frac{n}{5}} \le C(1+t)^{-21/20} X_m^2(t)^2. \tag{3.34}
$$

Substituting (3.34) into (3.32), we obtain

$$
\|Q_3(t)\|_{H^2} \le C(1+t)^{-21/20} X_m^2(t)^3. \tag{3.35}
$$

Similarly we can obtain (3.33). (3.33)–(3.35) shows (3.27). Proof of Lemma 3.2.2 is finished. ∎

Remark 3.2.1 Let the spatial dimension N be equal to 2. In general, if $m > 3l'/2 + 2$, $l' \in \mathbb{N}$, then we can obtain

$$
\begin{aligned}
\|JF(t)\|_{H^{l'-1}} &+ \sum_{\substack{|\alpha| = l' \\ |\beta| = 1}} \|(\partial^\alpha J^\beta F - P_{\alpha\beta})(t)\|_{L^2} \\
&\le C(1+t)^{-(1+4\varepsilon_1)} \Big(\|u(t)\|_{H^m} + \|Ju(t)\|_{H^{l'}}\Big)^3
\end{aligned} \tag{3.36}
$$

$$
\text{for} \quad u \in C\Big([0,T]; H^m\Big) \cap C\Big([0,T]; H^{l',1}\Big) \quad \text{satisfying}
$$

$$
\sup_{t \in [0,T]} \Big(\|u(t)\|_{H^m} + \|Ju(t)\|_{H^{l'}}\Big) \le R \quad \text{with some} \quad R > 0,
$$

$$
\text{where} \quad \varepsilon_1 = \frac{m - 3l'/2 - 2}{8(m-2)} > 0.
$$

3.3 L^p–L^q estimates and quadratic semilinear equations

In this section we prepare the decay estimates to prove the global existence theorem for quadratic semilinear Schrödinger equations. In this section we make strong use of the L^1–L^∞ estimate (3.8) of the fundamental solution $e^{it\Delta}$. It is easy to see that $e^{it\Delta}$ is a unitary group on $H^s(\mathbb{R}^N)$ ($s \in \mathbb{R}$), that is

$$\|e^{it\Delta}\phi\|_{H^s} = \|\phi\|_{H^s} \quad \text{for} \quad t \in \mathbb{R}. \tag{3.37}$$

Applying the Reisz–Thorin interpolation theorem (see, e.g., M. Reed and B. Simon textbook [41]) to (3.9) and (3.37), we obtain the L^p–L^q estimates

$$\|e^{it\Delta}\phi\|_{L^q} \le C_{q,N}|t|^{-N(1/p-1/2)}\|\phi\|_{L^p} \quad \text{for} \quad t \in \mathbb{R}\backslash\{0\}, \tag{3.38}$$

where (p, q) satisfies $1/p + 1/q = 1$ and $1 \le p \le 2$. On the other hand, making use of the Sobolev embeddings and (3.37), we have

$$\|e^{it\Delta}\phi\|_{L^q} \le C_{q,N}\|e^{it\Delta}\phi\|_{H^{s_1}} = C_{q,N}\|\phi\|_{H^{s_1}} \le C_{q,n}\|\phi\|_{W^{s_2,p}} \tag{3.39}$$

for $t \in \mathbb{R}$, where $s_1 = N(1/2 - 1/q)$ and $s_2 = [N(2/p - 1)] + 1$. Combining (3.38) and (3.39), we obtain

$$\|e^{it\Delta}\phi\|_{L^q} \le C_{q,N}(1 + |t|)^{-N(1/p-1/2)}\|\phi\|_{W^{s_2,p}} \quad \text{for} \quad t \in \mathbb{R}, \tag{3.40}$$

where p, q and s_2 satisfy $1/p + 1/q = 1$, $1 \le p \le 2$ and $s_2 = [N(2/p - 1)] + 1$.

We assume that $F(u, q)$ is quadratic. In this case, our time-decay estimates in the previous section is not suited to our method. The cubic nonlinear term has a structural nice property. In other words, a cubic nonlinear term is easy to treat in the sense that it is naturally adapted to our symbolic calculus. See Remark 2.3.1. In the present section we employ another time-decay estimates which can be available for the quadratic case. It was developed by S. Klainerman and G. Ponce ([32]), and J. Shatah ([43]) independently. Now it is a standard method to study the global existence of small solutions to a lot of evolutional semilinear partial differential equations, in which all of the coefficients of their principal parts are constant. Using their method, one can make effective use of the time-decay of fundamental solutions of various evolutional partial differential equations with constant coefficients. It is basically depends on the degree of the power of nonlinearity. In which class we obtain the time-decay estimates is decided by the power. For instance, to put $p = 1$ would be the best choise in (3.40). The integral equation which is equivallent to (1.1)–(1.2), is written as

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}F(u(\tau), \nabla u(\tau))d\tau. \tag{3.41}$$

If we apply L^1–L^∞ estimate to (3.41), then we have

$$\begin{aligned} \|u(t)\|_\infty &\le C(1+t)^{-N/2}\|u_0\|_{W^{N+1,1}} + C\int_0^t (1+t-\tau)^{-N/2}\|F(u(\tau), \nabla u(\tau))\|_{W^{N+1,1}}d\tau \\ &\le C(1+t)^{-N/2}\|u_0\|_{W^{N+1,1}} + C\int_0^t (1+t-\tau)^{-N/2}\|u(\tau)\|^2_{H^{N+2}}d\tau \end{aligned}$$

for $t \ge 0$. Even if $\|u(t)\|_{H^{N+1}}$ is bounded for $t \ge 0$, we cannot get the time-decay of solution u. Certainly, we could choose the exponent p as small as possible. But if p is smaller than some number decided by the power of nonlinearity, it is difficult to make good use of the time-decay of the fundamental solution $e^{it\Delta}$. According to [32] or [43], $p = 3/4$ and $q = 4$ is the best choise of the exponents in the above case. Namely we employ the estimate

$$\|e^{it\Delta}\phi\|_{L^4} \le C(1 + |t|)^{-N/4}\|\phi\|_{[N/2]+1,4/3} \quad \text{for} \quad t \in \mathbb{R} \tag{3.42}$$

for the quadratic semilinear equations. More precisely, $p = 2\rho/(2\rho - 1)$ and $q = 2\rho$ is the best exponents when the power of nonlinearity is equal to ρ.

In order that our linear estimates can be applicable to the quadratic semilinear equations, we need to solve (1.1)–(1.2) in the class of $H^m \cap H^{m-2,2}$. We prepare

Lemma 3.3.1 *We assume that the nonlinear term $F(u, q)$ is quadratic:*

$$F(u, q) = O(|u|^2 + |q|^2) \quad \textit{near} \quad (u, q) = 0.$$

Let m be a sufficiently large integer. Then

$$\begin{aligned} &\|F(t)\|_{W^{m-1,4/3}} + \|F(t)\|_{H^{m-1}} + \sum_{|\alpha|=m} \|(\partial^\alpha F - P_\alpha)(t)\|_{L^2} \\ \le\ & C(1+t)^{-N/4}\left(\tilde{X}^N_m(t)\right)^2, \qquad (3.43) \\ &\|JF(t)\|_{W^{m-3,4/3}} + \|JF(t)\|_{H^{m-3}} + \sum_{\substack{|\alpha|=m-2 \\ |\beta|=1}} \|(\partial^\alpha J^\beta F - P_\alpha)(t)\|_{L^2} \\ \le\ & C(1+t)^{-N/4+1}\left(\tilde{X}^N_m(t)\right)^2, \qquad (3.44) \\ &\|J^2F(t)\|_{W^{m-5,4/3}} + \|J^2F(t)\|_{H^{m-5}} + \sum_{\substack{|\alpha|=m-4 \\ |\beta|=2}} \|(\partial^\alpha J^\beta F - P_\alpha)(t)\|_{L^2} \\ \le\ & C(1+t)^{-N/4+2}\left(\tilde{X}^N_m(t)\right)^2, \qquad (3.45) \end{aligned}$$

$$\textit{for} \quad u \in \bigcap_{j=0}^{2} C([0, T]; H^{m-2j,j}) \quad \textit{and} \quad t \in [0, T],$$

where

$$
\begin{aligned}
\tilde{X}_m^N(t) &= \sum_{\substack{|\alpha|+2|\beta|\le m \\ |\beta|\le 2}} \|\partial^\alpha J^\beta u(t)\|_{L^2} \\
&\quad + \sum_{\substack{|\alpha|+2|\beta|\le m-[N/2]-2 \\ |\beta|\le 2}} (1+t)^{N/4-|\beta|}\|\partial^\alpha J^\beta u(t)\|_{L^4}.
\end{aligned}
$$

Proof. We remark here that the Sobolev embedding

$$
\|u\|_{L^\infty} \le C\|u\|_{W^{[N/4]+1,4}} \quad \text{for} \quad u \in W^{[N/4]+1,4}(\mathbb{R}^N). \tag{3.46}
$$

In the same way as Lemma 3.2.1, we can obtain the estimates of nonlinear term in L^2-type spaces. For instance, using the Gagliardo–Nirenberg inequality (3.1) and the Sobolev embeddings (3.46), we get

$$
\begin{aligned}
\|u(t)^2\|_{H^m} &\le C\|u(t)\|_{L^\infty}\||D|^m u(t)\|_{L^2} \\
&\le C\|u(t)\|_{W^{[N/4]+1,4}}\||D|^m u(t)\|_{L^2} \\
&= C(1+t)^{-N/4}\Big((1+t)^{N/4}\|u(t)\|_{W^{[N/4]+1,4}}\Big)\||D|^m u(t)\|_{L^2} \\
&\le C(1+t)^{-N/4}\tilde{X}_m^N(t)^2.
\end{aligned}
$$

Applying the calculation like the above one to $F(t)$, $JF(t)$ and $J^2F(t)$, we can obtain the L^2-type estimates in (3.43), (3.44) and (3.45).

On the other hand, we get the estimates in $L^{4/3}$-L^4-type spaces by the Hölder inequality with the exponents $3/4 = 1/2 + 1/4$. For instance, we have

$$
\begin{aligned}
\|u(t)^2\|_{L^{4/3}} &\le \|u(t)\|_{L^4}\|u(t)\|_{L^2} \\
&\le (1+t)^{-N/4}\Big((1+t)^{N/4}\|u(t)\|_{L^4}\Big)\|u(t)\|_{L^2} \\
&\le (1+t)^{-N/4}\tilde{X}_m^N(t)^2.
\end{aligned}
$$

Applying the calculation like the above one to $F(t)$, $JF(t)$ and $J^2F(t)$, we can obtain the $L^{4/3}$-L^4-type estimates in (3.43), (3.44) and (3.45). ∎

3.4 Parabolic regularization

In this section we are concerned with the parabolic regularization of our semilinear Schrödinger equations. More precisely, we present the parabolic regularization of (1.1)–(1.2) and the existence theorem of it. This is one of the preliminary lemmata to prove the local existence theorem for (1.1)–(1.2). Let us consider the initial value problem

$$
\begin{aligned}
\partial_t u^\varepsilon - (i+\varepsilon)\Delta u^\varepsilon &= F(u^\varepsilon, \nabla u^\varepsilon) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \qquad (3.47)\\
u^\varepsilon(0,x) &= u_0(x) \qquad \text{in} \quad \mathbb{R}^N, \qquad (3.48)
\end{aligned}
$$

where $\varepsilon \in (0,1)$. We remark here that the initial data u_0 is independent of $\varepsilon \in (0,1)$. As far as we consider (3.47)–(3.48) in the Sobolev space H^m, it is the same as the initial value problem for semilinear heat equations

$$
\begin{aligned}
\partial_t u^\varepsilon - \varepsilon\Delta u^\varepsilon &= F(u^\varepsilon, \nabla u^\varepsilon) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \\
u^\varepsilon(0,x) &= u_0(x) \qquad \text{in} \quad \mathbb{R}^N,
\end{aligned}
$$

in the sense of the regularity of solutions or estimates, because $e^{it\Delta}$ is a unitary group on H^m. In our case, the parabolic regularization overcome the loss of derivatives because the heat kernel $e^{-t\varepsilon\Delta}$ gains the regularity of order 1. Then we can easily solve the initial value problem (3.47)–(3.48).

Lemma 3.4.1 *Let m be an integer $> N/2+1$. Then for any $u_0 \in H^m$, there exists a time $T_\varepsilon = T(\varepsilon, \|u_0\|_{[N/2]+2})$ such that the initial value problem (3.47)–(3.48) possesses a unique solution $u^\varepsilon \in C\big([0,T_\varepsilon); H^m\big)$. Moreover the map $u_0 \in H^m \longmapsto u^\varepsilon \in C\big([0,T_\varepsilon); H^m\big)$ is continuous.*

Proof. Let $S^\varepsilon(t)$ be a semigroup generated by $\partial_t - (i+\varepsilon)\Delta$. (3.47)–(3.48) is equivalent to

$$
u^\varepsilon(t) = S^\varepsilon(t)u_0 + \int_0^t S^\varepsilon(t-\tau)F(u^\varepsilon(\tau), \nabla u^\varepsilon(\tau))d\tau.
$$

Then, we consider the nonlinear map $u \longmapsto \Phi^\varepsilon(u)$ defined by

$$
\Phi^\varepsilon(u)(t) = S^\varepsilon(t)u_0 + \int_0^t S^\varepsilon(t-\tau)F(u(\tau), \nabla u(\tau))d\tau.
$$

To solve (3.47)–(3.48), we have only to find a unique fixed point of the map Φ^ε in

$$
B_T^m = \Big\{ u \in C\big([0,T]; H^m\big) \;\Big|\; \max_{t\in[0,T]} \|u(t)\|_{H^m} \le 2\|u_0\|_{H^m} \Big\},
$$

with some $T > 0$. Clearly B_T^m is a closed subset of $C([0,T]; H^m)$ and then it is a complete metric space. Lemma 3.1.2 yields

$$
\begin{aligned}
\|F(u(t), \nabla u(t))\|_{H^{m-1}} &\le A_m\Big(\|u_0\|_{H^{[N/2]+2}}\Big)\|u_0\|_{H^m}, \\
\|F(u(t), \nabla u(t)) - F(v(t), \nabla v(t))\|_{H^{m-1}} &\le A_m\Big(\|u_0\|_{H^m}\Big)\|u(t)-v(t)\|_{H^m}
\end{aligned}
$$

for $u, v \in B_T^m$ and $t \in [0,T]$, where $A_m(\cdot)$ is a non-decreasing function on $[0,\infty)$. Using the Plancherel formula, we have

$$
\|\Phi^\varepsilon(u)(t)\|_{H^m} \le \|u_0\|_{H^m} + \int_0^t \|S^\varepsilon(t-\tau)F(u(\tau), \nabla u(\tau))\|_{H^m} d\tau
$$

$$
\begin{aligned}
&= \|u_0\|_{H^m} + \int_0^t \|\langle\xi\rangle^m e^{-(i+\varepsilon)(t-\tau)|\xi|^2}\mathcal{F}[F(u(\tau),\nabla u(\tau))]\|_{L^2} d\tau \\
&\le \|u_0\|_{H^m} + \int_0^t \sup_{\xi\in\mathbb{R}^N}\left(e^{-\varepsilon(t-\tau)|\xi|^2}\right)\|\langle\xi\rangle^{m-1}\mathcal{F}[F(u(\tau),\nabla u(\tau))]\|_{L^2} d\tau \\
&\le \|u_0\|_{H^m} + \int_0^t \frac{C}{\sqrt{\varepsilon(t-\tau)}}\|F(u(\tau),\nabla u(\tau))\|_{H^{m-1}} d\tau \\
&\le \|u_0\|_{H^m}\left(1+\frac{C}{\sqrt{\varepsilon}}A_m\big(\|u_0\|_{H^{[N/2]+2}}\big)T^{1/2}\right),
\end{aligned}
$$

for $u \in B_T^m$ and $t \in [0,T]$. This shows $\Phi^\varepsilon(u) \in C([0,T];H^m)$. Similarly we have

$$
\|\Phi^\varepsilon(u)(t)-\Phi^\varepsilon(v)(t)\|_{H^m} \le \frac{C}{\sqrt{\varepsilon}}A_m\big(\|u_0\|_{H^m}\big)T^{1/2}\max_{t\in[0,T]}\|u(t)-v(t)\|_{H^m},
$$

for $u, v \in B_T^m$ and $t \in [0,T]$. If we choose $T>0$ as

$$
\gamma = \frac{C}{\sqrt{\varepsilon}}A_m\big(\|u_0\|_{H^m}\big)T^{1/2} < 1,
$$

then we have $u \in B_T^m \longmapsto \Phi^\varepsilon(u) \in B_T^m$ and

$$
\max_{t\in[0,T]}\|\Phi^\varepsilon(u)(t)-\Phi^\varepsilon(v)(t)\|_{H^m} \le \gamma \max_{t\in[0,T]}\|u(t)-v(t)\|_{H^m}
$$

with some positive constant $\gamma < 1$. Hence the contraction mapping theorem implies that there exists a unique fixed point $u^\varepsilon \in B_T^m$ of the nonlinear map Φ^ε.

Let $T_{\varepsilon,m} > 0$ be the maximal existence time in H^m. Especially we write $T_{\varepsilon,[N/2]+2}$ as T_ε. In the same way as the estimate of $\Phi^\varepsilon(u)$, we get

$$
\|u^\varepsilon(t)\|_{H^m} \le \|u_0\|_{H^m}\frac{C}{\sqrt{\varepsilon}}A_m\Big(\max_{t\in[0,T_\varepsilon-s]}\|u^\varepsilon(t)\|_{H^{[N/2]+2}}\Big)\int_0^t (t-\tau)^{-1/2}\|u^\varepsilon(t)\|_{H^m} d\tau,
$$

for $t \in [0, T_\varepsilon - s]$ with an arbitrary $s > 0$. The Gronwall inequality implies

$$
\|u^\varepsilon(t)\|_{H^m} \le \|u_0\|_{H^m}\exp\left\{\frac{C}{\sqrt{\varepsilon}}A_m\Big(\max_{t\in[0,T_\varepsilon-s]}\|u^\varepsilon(t)\|_{H^{[N/2]+2}}\Big)(T_\varepsilon-s)^{1/2}\right\}
$$

for $t \in [0, T_\varepsilon - s]$ with an arbitrary $s > 0$. This shows that $T_{\varepsilon,m} = T_\varepsilon$, in other words $T_{\varepsilon,m}$ depends on $\|u_0\|_{H^{[N/2]+2}}$ and not on $\|u_0\|_{H^m}$, $m > [N/2]+2$.

We can show the continuity of the map $u_0 \longmapsto u^\varepsilon$ by the same technique used for the estimate on $\Phi^\varepsilon(u)(t) - \Phi^\varepsilon(v)(t)$. ∎

Chapter 4

Local existence for semilinear Schrödinger equations

4.1 Introduction to local existence theorems

This chapter is devoted to studying local existence for semilinear Schrödinger equations. As we mentioned in Chapter 1, these equations cannot allow the classical energy estimates of solutions. In other words, the loss of derivetives occurs. Then we have to resolve it in order to solve the initial value problem (1.1)–(1.2). For this purpose, we make strong use of the modification of the theory of linear Schrödinegr-type equations developed in Chapter 2.

In the present section we will introduce the history of the studies on the initial value problem for semilinear Schrödinger equations or nonlinear dispersive-type equations. But we omit here the history of the studies on the global existence of small solutions to semilinear Schrödinegr equations (see the next chapter). This subject began with the study on the initial value problem for Korteweg–de Vries equation in the middle of 1970' (see [1], [42] and [28] for instance), because the dispersive property (see, e.g., (1.6)) attracts a lot of attention of scientists. After these works, this subject developed into the studies on the initial value problems for nonlinear dispersive-type equations appearing in mathematical physics. These equations allow the classical energy estimates of solutions, and then a lot of researchers have been studying these problems from a viewpoint of the theory of evolution equations. More precisely, they studied local and global existence theorems, blow-up theorems, scattering, stability of solitary waves and etc (see, e.g., W. A. Strauss' lecture note [45] and its references).

Because "nonlinear Schrödinger equations" in mathematical physics is one of the typical examples of nonlinear dispersive-type partial differential equations, many authors studied the initial value problem for semilinear Schrödinger equations of the form

$$
\partial_t u - i\Delta u = f(u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \tag{4.1}
$$

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N. \tag{4.2}$$

where the nonlinear term $f(u)$ is independent of ∇u and satisfies $f(u) = -ig(|u|^2)u$, and $g(\cdot)$ is a real-valued function on $[0, +\infty)$. It is the generalization of $f(u) = \pm|u|^2u$ (see (1.3)). We remark here that these assumption leads the conservation of L^2-norm of solutions to (4.1)–(4.2), that is $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. In fact, multiplying (4.1) by $\bar{u}$ and integrating its real part over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^N} |u(t)|^2 dx = 0.$$

Let $G(\cdot)$ be the primitive of $g(\cdot)$ and satisfy $G(0) = 0$. Namely we define $G(s)$ by

$$G(s) = \int_0^s g(s')ds' \quad s \in [0, +\infty).$$

Then we have

$$\frac{\partial}{\partial \bar{u}} G(|u|^2) = \frac{dG}{ds}(|u|^2)\frac{\partial}{\partial \bar{u}}|u|^2 = g(|u|^2)u = if(u).$$

We multupuly (4.1) by $\partial_t\bar{u}$. Its twice imaginary part is

$$-\Delta u\partial_t\bar{u} - \Delta\bar{u}\partial_t u + g(|u|^2)(u\partial_t\bar{u} + \bar{u}\partial_t u). \tag{4.3}$$

Integrating (4.3) over $\mathbb{R}^N$, We have

$$\frac{d}{dt}E(u(t)) = 0, \quad \text{where} \quad E(u) = \int_{\mathbb{R}^N} |\nabla u|^2 + G(|u|^2)dx.$$

$E(u(t))$ is called a energy function.

Local existence theorem for (4.1)–(4.2) is proved by the contraction mapping theorem (see, e.g., [29] or [45, Chapter 3. Existence]). Using $E(u(t))$, many authors studied global existence or blow-up of H^1 or H^2-solutions to (4.1)–(4.2) ([17], [49], [29] and [18]). Roughly speaking, these results assert that the initial value problem (4.1)–(4.2) has a unique global solution if and only if the convervation laws $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and $E(u(t)) = E(u_0)$ give the *a priori* estimates of solution via the Sobolev embeddings. In other words, if $G(|u|^2)$ is negative and its absolute value is large, then one cannot get the *a priori* estimates of solution. Moreover, when the global existence of solutions was ensured under some assumption on the nonlinear term $f(u)$, some authors studied the scattering problem. In this subject, the solution $u(t)$ is approximated by $e^{it\Delta}u_{0,+}$ and $e^{it\Delta}u_{0,-}$ as $t \to \pm\infty$ respectively. They call the mapping from $u_{0,-}$ to $u_{0,+}$ the scattering operator. In all of the above works, they delt with the initial value problem (4.1)–(4.2) as that of ordinary differential equations which were valued some infinitely dimensional Hilbert spaces. We can say that the above mentioned works are based on the theory of evolution equations and are some branches of studies on nonlinear evolution equations: nonlinear wave equations, nonlinear Klein–Gordon equations, Yang–Mills equations, and etc. In other words, the theory of partial differential equations is not very applicable to these works.

Recently, however, some authors studied the initial value problem (1.1)–(1.2) or some semilinear Schrödinger equations and systems appearing in mathematical physics. As we mentioned in Chapter 1, these equations cannot be treated by the theory of evolution equations because they cause the loss of derivatives.

In [30], first, C. E. Kenig, G. Ponce and L. Vega analized the sharp smoothing effect (which they called) of $e^{it\Delta}$ and applied it to solving (1.1)–(1.2). Their analysis is based on the Fourier analysis. Let (τ, ξ) be the dual valuable of (t, x) under the Fourier transformation. They proved that the oprators whose symbols are $\xi_j/(\tau - |\xi|^2)$ $(j = 1, \cdots, N)$, are bounded in a sense. Using this fact and the method of the Neumann series, they constructed the inverse of Schrödinger-type operators

$$S = \partial_t - i\Delta + \sum_{j=1}^N b_j(t,x)\partial_j.$$

Clearly, in their work, the smallness of the coefficients $b_j(t,x)$ $(j = 1, \cdots, N)$ is essential to construct the inverse of S. In terms of semilinear equations, the smallness of the coefficients means that of the solutions to semilinear equations, because $b_1(t,x), \cdots, b_N(t,x)$ correspond to $\dfrac{\partial F}{\partial q_1}(u, \nabla u), \cdots, \dfrac{\partial F}{\partial q_N}(u, \nabla u)$. Then they proved the local existence of *small* solutions to (1.1)–(1.2). Moreover, F. Linares and G. Ponce ([34]) applied this sharp smoothing estimates to solving the elliptic–hyperbolic and hyperbolic–hyperbolic Davey–Stewartson equations. In general, such a method like the above gives only the local existence of *small* solutions (see also [10]).

Secondly, A. Soyeur ([44]) studied the local and the global existence for the Ishimori equation.

$$\begin{aligned}
\partial_t u - i(\partial_1^2 - \partial_2^2)u &= f_1(u, \partial_1 u, \partial_2 u) + f_2(u, \partial_1 u, \partial_2 u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \qquad (4.4)\\
f_1(u, \partial_1 u, \partial_2 u) &= \frac{-2i\bar{u}}{1+|u|^2}\Big((\partial_1 u)^2 - (\partial_2 u)^2\Big),\\
f_2(u, \partial_1 u, \partial_2 u) &= b\Big(\partial_1\varphi\partial_2 u + \partial_2\varphi\partial_1 u\Big),\\
(\partial_1^2 + \partial_2^2)\varphi &= 4i\frac{\partial_1 u\partial_2\bar{u} - \partial_1\bar{u}\partial_2 u}{(1+|u|^2)^2},
\end{aligned}$$

where b is a real constant. Because $f_2(u, \partial_1 u, \partial_2 u)$ does not give the loss of derivatives, (4.4) can be seen the same as the Heisenberg ferro-magnetic model (1.11) from our viewpoint. These equations are the typical examples of (2.10). Then he solved the initial value problem for (4.4) by means of some quantity which played a role of H^s-norm and corresponded to the gauge transformation with (2.12) (see [44, Section 2]).

In this chapter we present usual local existence theorems for the initial value problem (1.1)–(1.2) by using the linear estimates developed in Chapter 2 and well-known technique for nonlinear partial differential equations. which is prepared in Chapter 3. Section 4.2 is devoted to studying (1.1)–(1.2) in one space dimension. In Section 4.3 we prove the local existence theorem in general space dimensions. Section 4.2 can be seen as a part of Section 4.3. In Section 4.2. however, we make good use of the speciality of one dimensioanl case. More precisely, the loss of derivetives can be overcome by the gauge transformation developed in Section 2.2. On the other hand. in general dimensional case. we need the symbolic calculus of pseudo-diffrential operators appeared in Section 2.3. Then. one dimensional case is simpler than the general dimensional case.

4.2 Local existence for semilinear Schrödinger equations in one space dimension

In this section we present the local existence theorem for (1.1)–(1.2) in one space dimension. Throughout of the present section. we assume $N=1$. Our strategy consists of parabolic regularization in Section 3.4 and the uniform estimates of solutions to these equations. The later is obtained by the linear estimates developed in Section 2.2. Then the standard compactness argument implies the local existence theorem. In view of the nessecary condition of L^2-wellposedness for linear Schrödinger-type equations, our results devide into the quadratic case ($\rho=2$) and the cubic case ($\rho \geq 3$). Our results are the following.

Theorem 4.2.1 *We assume the spatial dimension N is equal to 1.*
(i) *Let $\rho \geq 3$ and let m be an integer which is greater than 3. Then. for any $u_0 \in H^m$, there exists a time $T = T\left(\|u_0\|_{H^3}\right) > 0$ such that the initial value problem* (1.1)–(1.2) *possesses a unique solution*

$$u \in C\left([0,T); H^m\right).$$

(ii) *Let $\rho = 2$ and let m be an integer which is greater than 3. Then. for any $u_0 \in H^m \cap H^{0,1}$. there exists a time $T = T\left(\|u_0\|_{H^3} + \|u_0\|_{H^{0,1}}\right) > 0$ such that the initial value problem* (1.1)–(1.2) *possesses a unique solution*

$$u \in C\left([0,T); H^m \cap H^{0,1}\right).$$

Proof of Part (i) *of Theorem* 4.2.1. We assume that $N=1$ and $\rho \geq 3$. We remark here that

$$\frac{\partial F}{\partial u}(u,q),\ \frac{\partial F}{\partial \bar{u}}(u,q),\ \frac{\partial F}{\partial q}(u,q),\ \frac{\partial F}{\partial \bar{q}}(u,q) = O\left(|u|^2+|q|^2\right) \quad \text{near} \quad (u,q)=0.$$

Existence. Let $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ be solutions to (3.47)–(3.48). We will show that there exists

a common time $T = T\left(\|u_0\|_{H^3}\right) > 0$ such that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$. We evaluate the following quantity

$$N_1^m(u^\varepsilon(t)) = \|u^\varepsilon(t)\|_{H^{m-1}} + \|k_1(t,\cdot;u^\varepsilon)v_m^\varepsilon(t)\|,$$

where

$$\begin{aligned}
k_1(t,x;u^\varepsilon) &= \exp\left(p_1(t,x;u^\varepsilon)\right)I, \\
p_1(t,x;u^\varepsilon) &= -\frac{1}{2}\int_0^x \operatorname{Im}\frac{\partial F}{\partial q}(u^\varepsilon,\partial u^\varepsilon)(t,y)dy, \\
v_m^\varepsilon(t) &= {}^t\left(\partial^m u^\varepsilon(t), \partial^m \overline{u^\varepsilon(t)}\right).
\end{aligned}$$

It is easy to see that

$$|p_1(t,x;u)| \leq C_1\|u(t)\|_{H^1}^2$$

for any $u \in C\left([0,T];H^1\right)$. We remark here that the positive constant C_1 is independent of $\varepsilon\in(0,1]$. In view of (2.17), $N_1^m(u^\varepsilon(t))$ is equivalent to $\|u^\varepsilon(t)\|_{H^m}$, that is

$$e^{-C_1\|u^\varepsilon(t)\|_{H^1}^2}\|u^\varepsilon(t)\|_{H^m} \leq N_1^m(u^\varepsilon(t)) \leq 2e^{C_1\|u^\varepsilon(t)\|_{H^1}^2}\|u^\varepsilon(t)\|_{H^m}. \tag{4.5}$$

Scince $u^\varepsilon(0)=u_0$ is independent of $\varepsilon\in(0,1]$, $N_1^m(u^\varepsilon(0))$ is also independent of $\varepsilon\in(0,1]$. Then we put $\alpha_m = N_1^m(u^\varepsilon(0)) = N_1^m(u_0)$. (4.5) implies

$$e^{-C_1\|u_0\|_{H^1}^2}\|u_0\|_{H^m} \leq \alpha_m \leq 2e^{C_1\|u_0\|_{H^1}^2}\|u_0\|_{H^m}.$$

Let T_m^ε be a positive time defined by

$$T_m^\varepsilon = \sup\left\{\, T>0 \,\middle|\, N_1^m(u^\varepsilon(t)) \leq 2\alpha_m \quad \text{for} \quad t\in[0,T) \,\right\}.$$

The positivity of T_m^ε is ensured by Lemma 3.4.1.

Operating $\langle D\rangle^{m-1}$ on the equation (3.47), we have

$$\partial_t(\langle D\rangle^{m-1}u^\varepsilon) - (i+\varepsilon)\partial^2(\langle D\rangle^{m-1}u^\varepsilon) = \langle D\rangle^{m-1}F(u^\varepsilon,\partial u^\varepsilon).$$

Multiplying $\langle D\rangle^{m-1}\overline{u^\varepsilon}$ and integrating the real part over $\mathbb{R}$, we get

$$\begin{aligned}
&\frac{d}{dt}\|u^\varepsilon(t)\|_{H^{m-1}}^2 + 2\varepsilon\|\partial u^\varepsilon(t)\|_{H^{m-1}}^2 \\
\leq\ & 2\|F(u^\varepsilon(t),\partial u^\varepsilon(t))\|_{H^{m-1}}\|u^\varepsilon(t)\|_{H^{m-1}} \\
\leq\ & 2A_m\left(\|u^\varepsilon\|_{H^2}\right)\|u^\varepsilon(t)\|_{H^m}\|u^\varepsilon(t)\|_{H^{m-1}},
\end{aligned}$$

where we used (3.2) and the Sobolev embedding $H^2(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R})$. then we obtain

$$\begin{aligned}
\frac{d}{dt}\|u^\varepsilon(t)\|_{H^{m-1}} &\leq A_m\left(\|u^\varepsilon\|_{H^2}\right)\|u^\varepsilon(t)\|_{H^m} \\
&\leq A_m\left(\|u^\varepsilon\|_{H^2}\right)N_1^m(u^\varepsilon(t)) \quad \text{for} \quad t\in[0,T_m^\varepsilon],
\end{aligned} \tag{4.6}$$

where we made use of (4.5). $A_m(\cdot)$ means a incresing function on $[0, \infty)$ which depends only on m and $F(u, q)$. We express such functions by the same notation $A_m(\cdot)$.

Next we estimate $\{v_m^\varepsilon\}$. Simple calculation implies that v_m^ε satisfies the 2×2-system

$$\Big(I\partial_t - \varepsilon I\Delta + iH_1^\varepsilon(t)\Big)v_m^\varepsilon = f_m^\varepsilon, \tag{4.7}$$

$$\begin{aligned}
H_1^\varepsilon(t) &= \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} - i \begin{bmatrix} b_1^\varepsilon(t,x) & b_2^\varepsilon(t,x) \\ \overline{b_2^\varepsilon(t,x)} & \overline{b_1^\varepsilon(t,x)} \end{bmatrix} \partial, \\
b_1^\varepsilon(t,x) &= -\frac{\partial F}{\partial q}(u^\varepsilon, \partial u^\varepsilon)(t,x), \qquad b_2^\varepsilon(t,x) = -\frac{\partial F}{\partial \bar{q}}(u^\varepsilon, \partial u^\varepsilon)(t,x), \\
f_m^\varepsilon(t,x) &= {}^t\Big(f_{m,1}^\varepsilon(t,x), \overline{f_{m,1}^\varepsilon(t,x)}\Big), \\
f_{m,1}^\varepsilon(t,x) &= \Big(\partial^m F(u^\varepsilon, \partial u^\varepsilon) - P(u^\varepsilon, \partial u^\varepsilon, \partial\partial^m u^\varepsilon)\Big)(t,x).
\end{aligned}$$

We put $w_m^\varepsilon(t,x) = k_1(t,x;u^\varepsilon)v_m^\varepsilon(t,x)$ and $g_m^\varepsilon(t,x) = k_1(t,x;u^\varepsilon)f_m^\varepsilon(t,x)$. Multiplying (4.7) by $k_1(t,x;u^\varepsilon)$, we have

$$\Big(I\partial_t - \varepsilon I\Delta + i\tilde{H}_1^\varepsilon(t)\Big)w_m^\varepsilon = g_m^\varepsilon(t,x),$$

where

$$\begin{aligned}
\tilde{H}_1^\varepsilon(t) &= \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} - i \begin{bmatrix} \mathrm{Re}\, b_1^\varepsilon(t,x) - \varepsilon \mathrm{Im}\, b_1^\varepsilon(t,x) & b_2^\varepsilon(t,x) \\ \overline{b_2^\varepsilon(t,x)} & \mathrm{Re}\, b_1^\varepsilon(t,x) - \varepsilon \mathrm{Im}\, b_1^\varepsilon(t,x) \end{bmatrix} \partial \\
&- i \begin{bmatrix} c_1^\varepsilon(t,x) & c_2^\varepsilon(t,x) \\ \overline{c_2^\varepsilon(t,x)} & \overline{c_1^\varepsilon(t,x)} \end{bmatrix}, \\
c_1^\varepsilon(t,x) &= -(i+\varepsilon)\left(\frac{1}{2}\partial b_1^\varepsilon(t,x) + \frac{1}{4}(\mathrm{Im}\, b_1^\varepsilon(t,x))^2\right) \\
&+ \frac{1}{2} b_1\varepsilon(t,x) \mathrm{Im}\, b_1\varepsilon(t,x) + \frac{1}{2}\int_0^x \partial_t \mathrm{Im}\, b_1^\varepsilon(t,y)dy, \\
c_2^\varepsilon(t,x) &= \frac{1}{2}\mathrm{Im}\, b_1^\varepsilon(t,x) b_2^\varepsilon(t,x).
\end{aligned}$$

In the same way as (2.24), we have

$$\frac{d}{dt}\|w_m^\varepsilon(t)\| \le C_2 B_{b_1^\varepsilon}(t)\|w_m^\varepsilon(t)\| + \|g_m^\varepsilon(t)\|,$$

where

$$\begin{aligned}
B_{b_1^\varepsilon}(t) &= \sup_{x\in\mathbb{R}}\left|\int_0^x \mathrm{Im}\, \partial_t b_1^\varepsilon(t,y)dy\right| \\
&+ \sum_{j=1}^2\Big(\sup_{x\in\mathbb{R}}|b_j^\varepsilon(t,x)| + \sup_{x\in\mathbb{R}}|b_j^\varepsilon(t,x)|^2\Big).
\end{aligned}$$

We remark here that C_2 is a positive constant and is independent of $\varepsilon \in (0, 1]$. Since $b_1^\varepsilon(t,x)$ and $b_2^\varepsilon(t,x)$ are quadratic terms of u^ε, $\overline{u^\varepsilon}$, ∂u^ε and $\partial \overline{u^\varepsilon}$, the Sobolev embedding yields

$$\sum_{j=1}^2\Big(\sup_{x\in\mathbb{R}}|b_j^\varepsilon(t,x)| + \sup_{x\in\mathbb{R}}|b_j^\varepsilon(t,x)|^2\Big) \le A_3\Big(\|u^\varepsilon(t)\|_{H^3}\Big).$$

On the other hand, using the equation (3.47), we get

$$\begin{aligned}
&\int_0^x \mathrm{Im}\, \partial_t b_1^\varepsilon(t,y)dy \\
&= \int_0^x \mathrm{Im}\, \partial_t \frac{\partial F}{\partial q}(u^\varepsilon, \partial u^\varepsilon)(t,y)dy \\
&= \int_0^x \mathrm{Im}\left\{\frac{\partial^2 F}{\partial u \partial q}(u^\varepsilon, \partial u^\varepsilon)\partial_t u^\varepsilon\right\}dy \\
&+ \int_0^x \mathrm{Im}\left\{\frac{\partial^2 F}{\partial \bar{u} \partial q}(u^\varepsilon, \partial u^\varepsilon)\partial_t \overline{u^\varepsilon}\right\}dy \\
&+ \int_0^x \mathrm{Im}\left\{\frac{\partial^2 F}{\partial q^2}(u^\varepsilon, \partial u^\varepsilon)\partial_t \partial u^\varepsilon\right\}dy \\
&+ \int_0^x \mathrm{Im}\left\{\frac{\partial^2 F}{\partial q \partial \bar{q}}(u^\varepsilon, \partial u^\varepsilon)\partial_t \partial \overline{u^\varepsilon}\right\}dy \\
&= \int_0^x \mathrm{Im}\left[\frac{\partial^2 F}{\partial u \partial q}(u^\varepsilon, \partial u^\varepsilon)\Big\{(i+\varepsilon)\partial^2 u^\varepsilon + F(u^\varepsilon, \partial u^\varepsilon)\Big\}\right]dy \\
&+ \int_0^x \mathrm{Im}\left[\frac{\partial^2 F}{\partial \bar{u} \partial q}(u^\varepsilon, \partial u^\varepsilon)\Big\{(-i+\varepsilon)\partial^2 \overline{u^\varepsilon} + \overline{F(u^\varepsilon, \partial u^\varepsilon)}\Big\}\right]dy \\
&+ \int_0^x \mathrm{Im}\left[\frac{\partial^2 F}{\partial q^2}(u^\varepsilon, \partial u^\varepsilon)\Big\{(i+\varepsilon)\partial^3 u^\varepsilon + \partial F(u^\varepsilon, \partial u^\varepsilon)\Big\}\right]dy \\
&+ \int_0^x \mathrm{Im}\left[\frac{\partial^2 F}{\partial q \partial \bar{q}}(u^\varepsilon, \partial u^\varepsilon)\Big\{(-i+\varepsilon)\partial^3 \overline{u^\varepsilon} + \partial \overline{F(u^\varepsilon, \partial u^\varepsilon)}\Big\}\right]dy.
\end{aligned}$$

Then we have

$$\sup_{x\in\mathbb{R}}\left|\int_0^x \mathrm{Im}\, \partial_t b_1^\varepsilon(t,y)dy\right| \le A_3\Big(\|u^\varepsilon(t)\|_{H^3}\Big). \tag{4.8}$$

(3.3) gives

$$\|g_m^\varepsilon(t)\| \le A_m(\|u^\varepsilon(t)\|_{H^2})\|u^\varepsilon(t)\|_{H^m}.$$

Noting (4.5), we obtain

$$\frac{d}{dt}\|w_m^\varepsilon(t)\| \le A_m\Big(N_1^3(u^\varepsilon(t))\Big)N_1^m(u^\varepsilon(t)) \quad \text{for} \quad t \in [0, T_m^\varepsilon]. \tag{4.9}$$

Combining (4.6) and (4.9), we have

$$\frac{d}{dt}N_1^m(u^\varepsilon(t)) \le C'(\alpha_3)N_1^m(u^\varepsilon(t)) \quad \text{for} \quad t \in [0, T_m^\varepsilon],$$

where $C'(\alpha_3)$ is a positive constant dpending only on m and α_3. The Gronwall inequality yields

$$N_1^m(u^\varepsilon(t)) \le \alpha_m e^{C(\alpha_3)t} \quad \text{for} \quad t \in [0, T_m^\varepsilon].$$

This implies

$$2 \le \exp\Big(C'(\alpha_3)T_m^\varepsilon\Big) \quad \text{i.e.} \quad T_m^\varepsilon \ge \frac{\log 2}{C'(\alpha_3)} \equiv T > 0.$$

Then $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$.

In view of the equation $\partial_t u^\varepsilon = (i+\varepsilon)\partial^2 u^\varepsilon + F(u^\varepsilon, \partial u^\varepsilon)$, $\{\partial_t\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^{m-2})$. Then $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $C^{0,1}\Big([0,T];H^{m-2}\Big)$. The simple interpolation implies that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $C^{0,\beta}\Big([0,T];H^{m-2\beta}\Big)$ for any $\beta \in (0,1]$. The Rellich theorem and the Ascoli–Arzelà theorem show that there exists subsequance $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in L^\infty(0,T;H^m)$ such that

$$\begin{aligned} u^\varepsilon &\xrightarrow{\mathrm{w}^*} u && \text{in} \quad L^\infty(0,T;H^m) \quad (\text{as} \quad \varepsilon \downarrow 0), \\ u^\varepsilon &\longrightarrow u && \text{in} \quad C\Big([0,T];H^{m-\beta}_{\mathrm{loc}}\Big) \quad (\text{as} \quad \varepsilon \downarrow 0) \quad \text{for any} \quad \beta > 0. \end{aligned}$$

It is easy to see that u is a solution to (1.1)–(1.2).

Uniqueness. Let u and u' belong to $L^\infty(0,T;H^m)$ and be solutions to (1.1)–(1.2). We put $v = {}^t(u - u', \overline{u - u'})$. v satisfies the initial value problem

$$\begin{aligned} \Big(I\partial_t + iH_1(t;u,u')\Big)v &= 0 \quad \text{in} \quad (0,T)\times\mathbb{R}, \\ v(0,x) &= 0 \quad \text{in} \quad \mathbb{R}, \end{aligned}$$

where

$$\begin{aligned} H_1(t;u,u') &= \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} - i\begin{bmatrix} b_1(t,x;u,u') & b_2(t,x;u,u') \\ \overline{b_2(t,x;u,u')} & \overline{b_1(t,x;u,u')} \end{bmatrix}\partial \\ &\quad - i\begin{bmatrix} c_1(t,x;u,u') & c_2(t,x;u,u') \\ \overline{c_2(t,x;u,u')} & \overline{c_1(t,x;u,u')} \end{bmatrix}, \\ b_1(t,x;u,u') &= -\int_0^1 \frac{\partial F}{\partial q}(\theta u + (1-\theta)u', \theta\partial u + (1-\theta)\partial u')d\theta, \\ b_2(t,x;u,u') &= -\int_0^1 \frac{\partial F}{\partial \bar{q}}(\theta u + (1-\theta)u', \theta\partial u + (1-\theta)\partial u')d\theta, \\ c_1(t,x;u,u') &= -\int_0^1 \frac{\partial F}{\partial u}(\theta u + (1-\theta)u', \theta\partial u + (1-\theta)\partial u')d\theta, \\ c_2(t,x;u,u') &= -\int_0^1 \frac{\partial F}{\partial u}(\theta u + (1-\theta)u', \theta\partial u + (1-\theta)\partial u')d\theta. \end{aligned}$$

Because v belongs to the class $C\Big([0,T];H^2\Big) \cap C^1\Big([0,T];L^2\Big)$, (2.20) implies

$$\|v(t)\| = 0 \quad \text{for} \quad t \in [0,T].$$

Continuity in the time variable. Let u belong to $L^\infty(0,T;H^m)$ and be a unique solution to (1.1)–(1.2). We put $v = {}^t(\partial^{m-1}u, \partial^{m-1}\bar{u})$. We have only to show

$$v \in \Big(C\Big([0,T];H^1\Big)\Big)^2. \tag{4.10}$$

Clearly, v is a solution to

$$\Big(I\partial_t + iH_1(t;u)\Big)v = f(t,x;u) \quad \text{in} \quad (0,T)\times\mathbb{R}, \tag{4.11}$$

$$v(0,x) = v_0 \quad \text{in} \quad \mathbb{R}, \tag{4.12}$$

where

$$\begin{aligned} H_1(t;u) &= \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} - i\begin{bmatrix} b_1(t,x;u) & b_2(t,x;u) \\ \overline{b_2(t,x;u)} & \overline{b_1(t,x;u)} \end{bmatrix}\partial \\ b_1(t,x;u) &= -\frac{\partial F}{\partial q}(u,\partial u), \quad b_2(t,x;u) = -\frac{\partial F}{\partial \bar{q}}(u,\partial u), \\ f(t,x;u) &= {}^t\Big(f_1(t,x;u), f_2(t,x;u)\Big), \quad \overline{f_1(t,x;u)} = f_2(t,x;u), \\ f_1(t,x;u) &= \partial^{m-1}F(u,\partial u) - P(u,\partial u,\partial\partial^{m-1}u). \end{aligned}$$

Since

$$u \in L^\infty(0,T;H^m) \cap C\Big([0,T];H^{m-\beta}\Big) \quad \text{for any} \quad \beta > 0,$$

it follows that

$$\begin{aligned} b_1(t,x;u), b_2(t,x;u) &\in C\Big([0,T];\mathcal{B}^{m-1}\Big) \cap C^1\Big([0,T];\mathcal{B}^{m-3}\Big), \\ f(t,x;u) &\in \Big(L^\infty(0,T;H^1)\Big)^2. \end{aligned}$$

Then, Proposition 2.2.2 implies (4.10). ∎

Proof of Part (ii) *of Theorem* 4.2.1. Let the spatial dimension N equal to 1 and let the power of nonlinearity ρ equal to 2. In the same way as the proof of Part (i), we can prove Part (ii). But there are a few differences because of $\rho - 2$. In Part (i) the boundedness of the gauge transformation $v_m^\varepsilon \mapsto k_1(t,\cdot;u)v_m^\varepsilon$ follows from the boundedness of $\|u^\varepsilon(t)\|_{H^1}$. In Part (ii), however, this does not hold. To overcome this point, we introduce the weighted sobolev space $H^{0,1}$. This seems to be quite natural from a viewpoint of the nessecary condition for L^2-well-posedness for linear Schrödinger-type equations (see 2.3). It is not clear whether the weighted Sobolev space is nessecary to solve the initial value problem for quadratic semilinear Schrödinger equation or not (see Remark 4.2.1). We will show mainly the differences between the proofs.

Let $\{u^\varepsilon\}$ be solutions to (3.47)–(3.48). We estimate the following quantity

$$N_2^m(u^\varepsilon(t)) = \|u^\varepsilon(t)\|_{H^{m-1}} + \|k_1(t,\cdot;u^\varepsilon)v_m^\varepsilon(t)\| + \|k_2(t,\cdot;u^\varepsilon)v_{01}^\varepsilon(t)\|,$$

where

$$\begin{aligned}
k_1(t,x;u^\varepsilon) &= \exp\Big(-p_1(t,x;u^\varepsilon)\Big)I\\
p_1(t,x;u^\varepsilon) &= -\frac{1}{2}\int_0^x \operatorname{Im}\frac{\partial F}{\partial q}(u^\varepsilon,\partial u^\varepsilon)(t,y)dy,\\
v_m^\varepsilon(t,x) &= {}^t(\partial^m u^\varepsilon(t,x), \partial^m\overline{u^\varepsilon(t,x)}),\\
k_2(t,x;u^\varepsilon) &= \exp\Big(-p_2(t,x;u^\varepsilon)\Big)I\\
p_2(t,x;u^\varepsilon) &= -\frac{1}{2}\int_0^x\int_0^1 \operatorname{Im}\frac{\partial F}{\partial q}(\theta u^\varepsilon,\theta\partial u^\varepsilon)(t,y)d\theta dy,\\
v_{01}^\varepsilon(t,x) &= {}^t(xu^\varepsilon(t,x), \overline{xu^\varepsilon(t,x)}).
\end{aligned}$$

We remark here that there exist two constants $\gamma_1,\gamma_2\in\mathbb{C}$ such that

$$\begin{aligned}
\operatorname{Im}\frac{\partial F}{\partial q}(u,q) &= \frac{1}{2i}\Big(\gamma_1 u - \overline{\gamma_1}\bar{u} + \gamma_2 q - \overline{\gamma_2}\bar{q}\Big)\\
&\quad + \text{real valued quadratic term of } (u,q).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\Big|p_1(t,x;u)\Big|,\ \Big|p_2(t,x;u)\Big| &\le C\left(\|u(t)\|_{L^\infty} + \int_{-\infty}^{+\infty}|u(t,x)|dx + \|u(t)\|_{H^1}^2\right)\\
&\le C\left(\|u(t)\|_{H^1} + \left(\int_{-\infty}^{+\infty}\langle x\rangle^{-2}dx\right)^{1/2}\|\langle x\rangle u(t)\|_{L^2} + \|u(t)\|_{H^1}^2\right)\\
&\le C\Big(\|u(t)\|_{H^1} + \|xu(t)\|_{L^2} + \|u(t)\|_{H^1}^2\Big).
\end{aligned}$$

In view of (2.17), $N_2^m(u^\varepsilon(t))$ is equivalent to $\|u^\varepsilon(t)\|_{H^m} + \|xu^\varepsilon(t)\|_{L^2}$, that is

$$\begin{aligned}
&\exp\Big(-C_1\Big(\|u(t)\|_{H^1} + \|xu(t)\|_{L^2} + \|u(t)\|_{H^1}^2\Big)\Big)\Big(\|u^\varepsilon(t)\|_{H^m} + \|xu^\varepsilon(t)\|_{L^2}\Big)\\
\le\ & N_2^m(u^\varepsilon(t))\\
\le\ & 2\exp\Big(C_1\Big(\|u(t)\|_{H^1} + \|xu(t)\|_{L^2} + \|u(t)\|_{H^1}^2\Big)\Big)\Big(\|u^\varepsilon(t)\|_{H^m} + \|xu^\varepsilon(t)\|_{L^2}\Big), \quad (4.13)
\end{aligned}$$

where $C_1>0$ is independent of $\varepsilon\in(0,1]$. We put $\alpha_m = N_2^m(u_0)$. We define $T_m^\varepsilon>0$ by

$$T_m^\varepsilon = \sup\Big\{\ T>0\ \Big|\ N_2^m(u^\varepsilon(t))\le 2\alpha_m \quad \text{for} \quad t\in[0,T)\ \Big\}.$$

In the same way as (4.6), we have

$$\frac{d}{dt}\|u^\varepsilon(t)\|_{H^{m-1}} \le A_m\Big(\|u^\varepsilon\|_{H^2}\Big)N_2^m(u^\varepsilon(t)) \quad \text{for} \quad t\in[0,T_m^\varepsilon]. \tag{4.14}$$

$w_m^\varepsilon(t,x)$, $g_m^\varepsilon(t,x)$, $b_1(t,x;u^\varepsilon)$, $b_2(t,x;u^\varepsilon)$ and $B_{b_1}^\varepsilon(t)$ are the same as those of the proof of Part (i). Similarly, we get

$$\frac{d}{dt}\|w_m^\varepsilon(t)\| \le C_2 B_{b_1^\varepsilon}(t)\|w_m^\varepsilon(t)\| + \|g_m^\varepsilon(t)\| \quad \text{for} \quad t\in[0,T_m^\varepsilon].$$

It is easy to see

$$\sum_{j=1}^2\Big(\sup_{x\in\mathbb{R}}|b_j^\varepsilon(t,x)| + \sup_{x\in\mathbb{R}}|b_j^\varepsilon(t,x)|^2\Big) \le A_m\Big(\|u^\varepsilon(t)\|_{H^3}\Big),$$

$$\|g_m^\varepsilon(t)\| \le A_m\Big(\|u(t)\|_{H^2} + \|xu(t)\|_{L^2}\Big)N_2^m(u^\varepsilon(t)).$$

Simple calculation yields

$$\begin{aligned}
&\left|\int_0^x \operatorname{Im}\partial_t b_1(t,x;u^\varepsilon)(t,y)dy\right|\\
\le\ &\left|\int_0^x \frac{1}{2i}\Big(\gamma_1\partial_t u^\varepsilon - \overline{\gamma_1}\partial_t\overline{u^\varepsilon} + \gamma_2\partial_t\partial u^\varepsilon - \overline{\gamma_2}\partial_t\partial\overline{u^\varepsilon}\Big)(t,y)\right.\\
&\quad \left. +\partial_t\Big(\text{real valued quadratic term of } u \text{ and } \partial u\Big)(t,y)dy\right|\\
\le\ &\frac{1}{2}\Big|\gamma_2\partial_t u^\varepsilon(t,x) - \gamma_2\partial_t u^\varepsilon(t,0) - \overline{\gamma_2}\partial_t\overline{u^\varepsilon(t,x)} + \overline{\gamma_2}\partial_t\overline{u^\varepsilon(t,0)}\Big|\\
+\ &\left|\frac{1}{2i}\int_0^x\Big(\gamma_1 i\partial^2 u^\varepsilon + \overline{\gamma_1}i\partial^2\overline{u^\varepsilon} + \gamma_1 F(u^\varepsilon,\partial u^\varepsilon) - \overline{\gamma_1}\overline{F(u^\varepsilon,\partial u^\varepsilon)}\Big)(t,y)dy\right|\\
+\ &A_3\Big(\|u^\varepsilon(t)\|_{H^3}\Big)\\
\le\ &C\Big(|\partial^2 u^\varepsilon(t,x)| + |\partial^2 u^\varepsilon(t,0)| + |F(u^\varepsilon,\partial u^\varepsilon)(t,x)| + |F(u^\varepsilon,\partial u^\varepsilon)(t,0)|\Big)\\
+\ &C\Big(|\partial u^\varepsilon(t,x)| + |\partial u^\varepsilon(t,0)|\Big) + A_3\Big(\|u^\varepsilon(t)\|_{H^3}\Big)\\
\le\ &A_3\Big(\|u^\varepsilon(t)\|_{H^3}\Big).
\end{aligned}$$

Then we get

$$\frac{d}{dt}\|w_m^\varepsilon(t)\| \le A_m(N_2^3(u^\varepsilon(t)))N_2^m(u^\varepsilon(t)) \quad \text{for} \quad t\in[0,T_m^\varepsilon]. \tag{4.15}$$

On the other hand, v_{01}^ε satisfies

$$\Big(I\partial_t - \varepsilon I\Delta + iH_1'(t;u^\varepsilon)\Big)v_{01}^\varepsilon = f_{01}^\varepsilon, \tag{4.16}$$

$$\begin{aligned}
H_1'(t;u^\varepsilon) &= \begin{bmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{bmatrix} - i \begin{bmatrix} b_1'(t,x;u^\varepsilon) & b_2'(t,x;u^\varepsilon) \\ \overline{b_2'(t,x;u^\varepsilon)} & \overline{b_1'(t,x;u^\varepsilon)} \end{bmatrix} \partial, \\
&\quad - i \begin{bmatrix} c_1'(t,x;u^\varepsilon) & c_2'(t,x;u^\varepsilon) \\ \overline{c_2'(t,x;u^\varepsilon)} & \overline{c_1'(t,x;u^\varepsilon)} \end{bmatrix}, \\
b_1'(t,x;u^\varepsilon) &= -\int_0^1 \frac{\partial F}{\partial q}(\theta u^\varepsilon, \theta\partial u^\varepsilon)(t,x)d\theta, \\
b_2'(t,x;u^\varepsilon) &= -\int_0^1 \frac{\partial F}{\partial \bar{q}}(\theta u^\varepsilon, \theta\partial u^\varepsilon)(t,x)d\theta, \\
c_1'(t,x;u^\varepsilon) &= -\int_0^1 \frac{\partial F}{\partial u}(\theta u^\varepsilon, \theta\partial u^\varepsilon)(t,x)d\theta, \\
c_2'(t,x;u^\varepsilon) &= -\int_0^1 \frac{\partial F}{\partial \bar{u}}(\theta u^\varepsilon, \theta\partial u^\varepsilon)(t,x)d\theta, \\
f_{01}^\varepsilon(t,x) &= {}^t\Big(f_{01,1}^\varepsilon(t,x), \overline{f_{01,1}^\varepsilon(t,x)}\Big), \\
f_{01,1}^\varepsilon(t,x) &= -2i\partial u + b_1'(t,x;u^\varepsilon)u + b_2'(t,x;u^\varepsilon)\bar{u}.
\end{aligned}$$

In the same way as (4.15), we get

$$\frac{d}{dt}\|k_2(t,\cdot;u^\varepsilon)v_{01}^\varepsilon(t)\| \le A_m(N_2^3(u^\varepsilon(t)))N_2^m(u^\varepsilon(t)) \quad \text{for} \quad t \in [0, T_m^\varepsilon]. \tag{4.17}$$

Combining (4.14), (4.15) and (4.17), we obtain

$$\frac{d}{dt}N_2^\varepsilon(u^\varepsilon(t)) \le A_m(N_2^3(u^\varepsilon(t)))N_2^m(u^\varepsilon(t)) \quad \text{for} \quad t \in [0, T_m^\varepsilon].$$

This implies that there exists a common time $T = T\big(\|u_0\|_{H^3}+\|xu_0\|_{L^2}\big) > 0$ such that $\{u^\varepsilon\}$ is bounded in $L^\infty(0,T;H^m \cap H^{0,1})$. Then, we can get a solution $u \in L^\infty(0,T;H^m \cap H^{0,1})$ to (1.1)–(1.2) provided that $\varepsilon \downarrow 0$.

The uniquness and the continuity in the time variable can be proved by the same enrgy method. We omit the detail of the proofs of them. ∎

Remark 4.2.1 As we mentioned at the beginning of the proof of Part (ii) of Theorem 4.2.1, it is not known whether the weighted Sobolev space is needed to solve the initial value problem for the quadtatic semilinear equations. We expect that the local existence theorem for it does not hold in the usual Sobolev space H^m. Let us consider the following initial value problem

$$\begin{aligned}
\partial_t u - i\partial^2 u + i\partial(|u|^2) &= 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}, &(4.18)\\
u(0,x) &= u_0(x) \qquad \text{in} \quad \mathbb{R}. &(4.19)
\end{aligned}$$

Integrating (4.18) on $\mathbb{R}$, we have

$$\frac{d}{dt}\int_{-\infty}^{+\infty} u(t,x)dx = 0, \quad \text{i.e.} \quad \int_{-\infty}^{+\infty} u(t,x)dx = \int_{-\infty}^{+\infty} u_0(x)dx. \tag{4.20}$$

On the other hand, the gauge transformation related to (4.18)–(4.19) is

$$u(t,x) \longmapsto v(t,x) = u(t,x)\exp\Big(\frac{1}{2}\int_{-\infty}^{x} \mathrm{Re}\, u(t,y)dy\Big) \tag{4.21}$$

If the initial data u_0 belong to H^3 and satisfy

$$\Big|\int_{-\infty}^{+\infty} \mathrm{Re}\, u_0(x)\Big| = +\infty$$

(e.g., $u_0(x) = (1+i)\langle x\rangle^{-1/2}$), then (4.20) implies that

$$\Big|\int_{-\infty}^{x} \mathrm{Re}\, u(t,y)dy\Big| \longrightarrow +\infty \quad \text{as} \quad x \to +\infty.$$

Therefore the gauge transformation (4.21) cannot be an automorphic in L^2. Combining the idea of the necessary conditon of L^2-wellposedness and the above, we want to prove *non-existence* results in the usual Sobolev space. Unfortunately, however, we do not yet make sure this. For this purpose, we seem to need to know the pointwise behavior of $\mathrm{Im}\,\partial u(t,x)$ in detail, because

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{x} \mathrm{Re}\, u(t,y)dy &= \int_{-\infty}^{x} \partial_t \mathrm{Re}\, u(t,y)dy \\
&= \int_{-\infty}^{x} \mathrm{Re}\Big\{i\partial^2 u(t,y) - i\partial(|u(t,y)|^2)\Big\}dy \\
&= -\mathrm{Im}\,\partial u(t,x).
\end{aligned}$$

Remark 4.2.2 In the quadratic case (i.e. $\rho = 2$), we can replace $T = T\big(\|u_0\|_{H^3}+\|xu_0\|_{L^2}\big)$ by $T = T\big(\|u_0\|_{H^3} + \|J(t_0)u_0\|_{L^2}\big)$, if the initial time is $t_0 \in \mathbb{R}$. We remark here that $J(0) = x + 2it\partial$. This is very important when we consider the global existence theorem for quadratic semilinear equations.

4.3 Local existence for semilinear Schrödinger equations in genaral space dimensions

This section is concerned with the local existence of solutions to (1.1)–(1.2) in general space dimensions. Our strategy is basically same as that of the previous section which consists of the parabolic regularization and the uniform estimates. In the present section, however, we make strong use of pseudo-differential operators to get the uniform estimates. This is quite natural from the viewpoint of the theory of linear Schrödinger-type equations. In the same way as the previous section, our results devide in two parts according to the power of nonlinearity ρ. More precisely, we solve (1.1)–(1.2) in the usual Sobolev space H^m if $\rho \ge 3$, and in the weighted Sobolev space $H^m \cap H^{m-2,2}$ if $\rho = 2$. Our results are the following.

Theorem 4.3.1 *Let N be an arbitrary positive integer.*
(i) *We assume $\rho \geq 3$. Let m_1 be a sufficiently large integer. Then, for any $u_0 \in H^m$ ($m \in \mathbb{N} \geq m_1$), there exists a time $T = T\big(\|u_0\|_{H^{m_1}}\big) > 0$ such that the initial value problem (1.1)–(1.2) possesses a unique solution*

$$u \in C\big([0,T]; H^m\big).$$

(ii) *We assume $\rho = 2$. Let m_2 be a sufficiently large integer. Then, for any $u_0 \in H^m \cap H^{m-2,2}$, ($m \in \mathbb{N} \geq m_2$), there exists a time $T = T\big(\|u_0\|_{H^{m_1}} + \|u_0\|_{H^{m_2-2,2}}\big) > 0$ such that the initial value problem (1.1)–(1.2) possesses a unique solution*

$$u \in C\big([0,T]; H^m \cap H^{m-2,2}\big).$$

Remark 4.3.1 Since our analysis is based on the symbolic calculus of pseudo-differential operators, it is very troublesome to determine the minimum of m_1 and m_2 concretely.

Proof of Part (i) *of Theorem 4.3.1.*
Existence. Let $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ be solutions to (3.47)–(3.48). We show that there exists a common time $T > 0$ such that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$. For this purpose, we make use of the linear estimates developed in Section 2.3. For the sake of convenience, we introduce the following notations:

$$v_\alpha^\varepsilon = {}^t(\partial^\alpha u^\varepsilon, \partial^\alpha \overline{u^\varepsilon}), \quad v_{\alpha,0} = {}^t(\partial^\alpha u_0, \partial^\alpha \overline{u_0}), \qquad \text{for } |\alpha| = m,$$

$$\begin{aligned}
b_{11j}^\varepsilon(t,x) &= -\frac{\partial F}{\partial q_j}(u^\varepsilon, \nabla u^\varepsilon), \quad b_{12j}^\varepsilon(t,x) = -\frac{\partial F}{\partial \overline{q}_j}(u^\varepsilon, \nabla u^\varepsilon),\\
b_{21j}^\varepsilon(t,x) &= \frac{\overline{\partial F}}{\partial q_j}(u^\varepsilon, \nabla u^\varepsilon), \quad b_{22j}^\varepsilon(t,x) = \frac{\overline{\partial F}}{\partial q_j}(u^\varepsilon, \nabla u^\varepsilon),
\end{aligned}$$

$$\begin{aligned}
b^\varepsilon(t,x,\xi) &= \sum_{j=1}^N \begin{bmatrix} b_{11j}^\varepsilon(t,x) & b_{12j}^\varepsilon(t,x) \\ b_{21j}^\varepsilon(t,x) & b_{22j}^\varepsilon(t,x) \end{bmatrix} \xi_j,\\
b^{\varepsilon,\mathrm{diag}}(t,x,\xi) &= \sum_{j=1}^N \begin{bmatrix} b_{11j}^\varepsilon(t,x) & 0 \\ 0 & b_{22j}^\varepsilon(t,x) \end{bmatrix} \xi_j,
\end{aligned}$$

$$\begin{aligned}
K^\varepsilon(t) &= k^\varepsilon(t,x,D),\\
k^\varepsilon(t,x,\xi) &= \begin{bmatrix} e^{-p^\varepsilon(t,x,\xi)} & 0 \\ 0 & e^{p^\varepsilon(t,x,\xi)} \end{bmatrix},\\
p^\varepsilon(t,x,\xi) &= \sum_{j=1}^N \int_0^{x_j} \phi_j^\varepsilon(t,s)\, ds\, \xi_j \langle \xi_j \rangle^{-1},
\end{aligned}$$

$$\begin{aligned}
\phi_j^\varepsilon(t,x_j) &= M \sum_{|\alpha| \leq \left[\frac{N-1}{2}\right]+2} \int_{\mathbb{R}^{N-1}} |\partial^\alpha u^\varepsilon(t,x)|^2 d\check{x}_j \quad \text{with some} \quad M > 0,\\
\Lambda^\varepsilon(t) &= I + \tilde{\Lambda}^\varepsilon(t), \quad \tilde{\Lambda}^\varepsilon(t) = \tilde{\lambda}^\varepsilon(t,x,D),\\
\tilde{\lambda}^\varepsilon(t,x,\xi) &= \frac{1}{2}\sum_{j=1}^N \begin{bmatrix} 0 & b_{12j}^\varepsilon(t,x) \\ -b_{21j}^\varepsilon(t,x) & 0 \end{bmatrix} \xi_j \langle\xi\rangle^{-2},\\
N_3^m(u^\varepsilon(t)) &= \sum_{|\alpha|=m} \|K^\varepsilon(t)\Lambda^\varepsilon(t) v_\alpha^\varepsilon(t)\| + \|u^\varepsilon(t)\|_{H^{m-1}},\\
\alpha_m = N_3^m(u^\varepsilon(0)) &= \sum_{|\alpha|=m} \|K^\varepsilon(0)\Lambda^\varepsilon(0) v_{\alpha,0}\| + \|u_0\|_{H^{m-1}} \quad (\text{ind. of } \varepsilon \in (0,1]),\\
f_\alpha^\varepsilon &= {}^t(f_{\alpha,1}^\varepsilon, f_{\alpha,2}^\varepsilon),\\
f_{\alpha,1}^\varepsilon = \overline{f_{\alpha,2}^\varepsilon} &= P(u^\varepsilon, \nabla u^\varepsilon, \partial\partial^\alpha u^\varepsilon).
\end{aligned}$$

In the same way as in Section 2.3, we introduce

$$\begin{aligned}
B_{K^\varepsilon}(t) &= |e^{-p^\varepsilon(t,\cdot,\cdot)}|_l^{(0)} + |e^{p^\varepsilon(t,\cdot,\cdot)}|_l^{(0)} \geq 2,\\
B_{b^\varepsilon}(t) &= \sum_{m,n=1,2} \sum_{j=1}^N \sum_{|\alpha|\leq l} \Big(\sup_{x\in\mathbb{R}^N} |\partial^\alpha b_{mnj}^\varepsilon(t,x)| + \sup_{x\in\mathbb{R}^N} |\partial_t \partial^\alpha b_{mnj}^\varepsilon(t,x)| \Big),\\
B_{\phi^\varepsilon}^0(t) &= \sum_{j=1}^N \int_{-\infty}^{+\infty} \phi_j^\varepsilon(t,x_j)\,dx_j, \quad B_{\phi^\varepsilon}^1(t) = \sum_{j=1}^N \sup_{x_j\in\mathbb{R}} \left| \int_0^{x_j} \partial_t \phi_j^\varepsilon(t,y_j)\,dy_j \right|,\\
B_{\phi^\varepsilon}^\infty(t) &= \sum_{j=1}^N \sum_{k=0}^l \Big(\sup_{x_j\in\mathbb{R}} |\partial_j^k \phi_j^\varepsilon(t,x_j)| \Big).
\end{aligned}$$

We estimate $N_3^m(u^\varepsilon(t))$. Let T_m^ε be a positive time defined by

$$T_m^\varepsilon = \sup\Big\{ T > 0 \;\Big|\; N_3^m(u^\varepsilon(t)) \leq 2\alpha_m \quad \text{for} \quad t \in [0,T) \Big\}.$$

Lemma 3.4.1 ensures $T_m^\varepsilon > 0$. In view of the Sobolev embedding, there exists a constant $R = R(\alpha_{m_1}) > 0$ such that $\|u^\varepsilon(t)\|_{W^{1,\infty}} \leq R$ for $t \in [0, T_{m_1}^\varepsilon]$ and $\varepsilon \in (0,1]$. Moreover there exists a constant $C(m, \alpha_{m_1}) > 0$ such that

$$B_{b^\varepsilon}(t), B_{K^\varepsilon}(t), B_{\phi^\varepsilon}^0(t), B_{\phi^\varepsilon}^1(t), B_{\phi^\varepsilon}^\infty(t) \leq C(m, \alpha_{m_1}),$$

$$C(m,\alpha_{m_1})^{-1} N_3^m(u^\varepsilon(t)) \leq \|u^\varepsilon(t)\|_{H^m} \leq C(m,\alpha_{m_1}) N_3^m(u^\varepsilon(t)), \tag{4.22}$$

$$\|F(u^\varepsilon(t), \nabla u^\varepsilon(t))\|_{H^{m-1}} + \sum_{|\alpha|=m} \|K^\varepsilon(t)\Lambda^\varepsilon(t) f_\alpha^\varepsilon(t)\| \leq C(m,\alpha_{m_1}) N_3^m(u^\varepsilon(t)) \tag{4.23}$$

for $t \in [0, T_m^\varepsilon]$ and $\varepsilon \in (0,1]$, where we made use of (2.29), (2.30), (3.2) and (3.3). In the same way as (4.8), we obtain here only the estimates of $B_{\phi^\varepsilon}^1(t)$. Simple calculation yields

$$\int_0^{x_j} \partial_t \phi_j^\varepsilon(t,y_j)\,dy_j$$

$$
\begin{aligned}
&= M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_0^{x_j} \int_{\mathbb{R}^{N-1}} \partial_t |\partial^\alpha u^\varepsilon(t,y_j,\check{x}_j)|^2 d\check{x}_j dy_j \\
&= M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_0^{x_j} \int_{\mathbb{R}^{N-1}} \left(\partial_t\partial^\alpha u^\varepsilon \overline{\partial^\alpha u^\varepsilon} + \partial^\alpha u^\varepsilon \partial_t \overline{\partial^\alpha u^\varepsilon}\right)(t,y_j,\check{x}_j) d\check{x}_j dy_j \\
&= M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_0^{x_j} \int_{\mathbb{R}^{N-1}} \left(i\Delta\partial^\alpha u^\varepsilon \overline{\partial^\alpha u^\varepsilon} - \partial^\alpha u^\varepsilon \overline{i\Delta\partial^\alpha u^\varepsilon}\right)(t,y_j,\check{x}_j) d\check{x}_j dy_j \\
&+ M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_0^{x_j} \int_{\mathbb{R}^{N-1}} \left(\partial^\alpha F(u^\varepsilon,\nabla u^\varepsilon)\overline{\partial^\alpha u^\varepsilon} + \partial^\alpha u^\varepsilon \overline{\partial^\alpha F(u^\varepsilon,\nabla u^\varepsilon)}\right)(t,y_j,\check{x}_j) d\check{x}_j dy_j \\
&= M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_{\mathbb{R}^{N-1}} \left(i\partial_j\partial^\alpha u^\varepsilon \overline{\partial^\alpha u^\varepsilon} - \partial^\alpha u^\varepsilon \overline{i\partial_j\partial^\alpha u^\varepsilon}\right)(t,x_j,\check{x}_j) d\check{x}_j \\
&+ M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_{\mathbb{R}^{N-1}} \left(i\partial_j\partial^\alpha u^\varepsilon \overline{\partial^\alpha u^\varepsilon} - \partial^\alpha u^\varepsilon \overline{i\partial_j\partial^\alpha u^\varepsilon}\right)(t,0,\check{x}_j) d\check{x}_j \\
&+ M \sum_{|\alpha|\le[\frac{N-1}{2}]+2} \int_0^{x_j} \int_{\mathbb{R}^{N-1}} \left(\partial^\alpha F(u^\varepsilon,\nabla u^\varepsilon)\overline{\partial^\alpha u^\varepsilon} + \partial^\alpha u^\varepsilon \overline{\partial^\alpha F(u^\varepsilon,\nabla u^\varepsilon)}\right)(t,y_j,\check{x}_j) d\check{x}_j dy_j \\
&\le C \sup_{x_j\in\mathbb{R}} \sum_{|\alpha|\le[\frac{N-1}{2}]+3} \int_{\mathbb{R}^{N-1}} |\partial^\alpha u^\varepsilon(t,x)|^2 d\check{x}_j \\
&+ CA_{[\frac{N-1}{2}]+2}\left(\|u^\varepsilon(t)\|_{W^{1,\infty}}\right)\|u^\varepsilon(t)\|_{H^{[\frac{N-1}{2}]+2}}.
\end{aligned}
$$

Then we have $B^1_{\phi^\varepsilon}(t) \le A\left(\|u^\varepsilon(t)\|_{H^{m_1-1}}\right)$.

We choose the positive constant $M > 0$ satisfying

$$
\begin{aligned}
\left|b^\varepsilon_{nnj}(t,x)\right| &= \left|\operatorname{Im}\frac{\partial F}{\partial q_j}(u^\varepsilon(t),\nabla u^\varepsilon(t))\right| \\
&\le C\left(|u^\varepsilon(t,x)| + |\partial u^\varepsilon(t,x)|\right)^2 \\
&\le M\int_{\mathbb{R}^{N-1}}\left|\langle D\rangle^{\frac{N-1}{2}+2}u^\varepsilon(t,x)\right|^2 d\check{x}_j = \phi^\varepsilon_j(t,x_j),
\end{aligned}
$$

for any $\varepsilon \in (0,1]$ and any $t \in [0,T^\varepsilon_m]$.

v^ε_α satisfies

$$
\begin{gathered}
\left(I\partial_t + \varepsilon I|D|^2 + i\left(a(D) + b^\varepsilon(t,x,D)\right)\right)v^\varepsilon_\alpha = f^\varepsilon_\alpha \\
v^\varepsilon_\alpha(0,x) = v_{\alpha,0}(x).
\end{gathered}
$$

The local wellposedness for (3.47)–(3.48) ensures the validity of the energy estimates. We note here that

$$
\varepsilon\Lambda^\varepsilon(t)I|D|^2 = \varepsilon I|D|^2\Lambda^\varepsilon(t) + \varepsilon R^\varepsilon_{11}(t),
$$

$$
R^\varepsilon_{11}(t) = \tilde{\Lambda}^\varepsilon(t)|D|^2 - |D|^2\tilde{\Lambda}^\varepsilon(t), \quad \|R^\varepsilon_{11}(t)\|_{\mathcal{L}\left((L^2)^2\right)} \le C(m,\alpha_{m_1}) \quad \text{for} \quad t\in[0,T^\varepsilon_m].
$$

(2.42) implies

$$
\begin{gathered}
(I\partial_t + \varepsilon I|D|^2 + ia(D) + q^\varepsilon(t,x,D))K^\varepsilon(t)\Lambda^\varepsilon(t)v^\varepsilon_\alpha + R_{12,\varepsilon}(t)v^\varepsilon_\alpha = K^\varepsilon(t)\Lambda^\varepsilon(t)f^\varepsilon_\alpha, \\
q^\varepsilon(t,x,D) = 2(I + i\varepsilon\tilde{I})\sum_{j=1}^N \phi^\varepsilon_j(t,x_j)D_j^2\langle D_j\rangle^{-1} + b^{\varepsilon,\mathrm{diag}}(t,x,D), \\
\|R^\varepsilon_{12}(t)\|_{\mathcal{L}\left((L^2)^2\right)} \le C(m,\alpha_{m_1}) \quad \text{for} \quad t\in[0,T^\varepsilon_m].
\end{gathered}
$$

In the same way as (2.48), we get

$$
\frac{d}{dt}\|K^\varepsilon(t)\Lambda^\varepsilon(t)v^\varepsilon_\alpha(t)\| \le C(m,\alpha_{m_1})N^m_3(u^\varepsilon(t)) + \|K^\varepsilon(t)\Lambda^\varepsilon(t)f^\varepsilon_\alpha(t)\| \tag{4.24}
$$

On the other hand, we have

$$
\frac{d}{dt}\|u^\varepsilon(t)\|_{H^{m-1}} \le \|F(u^\varepsilon(t),\nabla u^\varepsilon(t))\|_{H^{m-1}}. \tag{4.25}
$$

Combining (4.23), (4.24) and (4.25), we obtain

$$
\frac{d}{dt}N^m_3(t) \le C(m,\alpha_{m_1})N^m_3(t), \quad \text{for} \quad t\in[0,T^\varepsilon_m].
$$

The Gronwall inequality yields

$$
N^m_3(t) \le \alpha_m \exp\left(C(m,\alpha_{m_1})t\right) \quad \text{for} \quad t\in[0,T^\varepsilon_m]. \tag{4.26}
$$

If we put $m = m_1$ and $t = T^\varepsilon_m$ in (4.26), then we have

$$
T^\varepsilon_{m_1} \ge T \equiv (\log 2)C(m,\alpha_{m_1})^{-1} > 0.
$$

Hence $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$ and then there exist a subsequence $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in L^\infty(0,T;H^m)$ such that

$$
u^\varepsilon \xrightarrow{\mathrm{W}^*} u \quad \text{in} \quad L^\infty(0,T;H^m) \quad \text{as} \quad \varepsilon\downarrow 0.
$$

The standard compactness argument implies that u is a solution to (1.1)–(1.2).

Uniqueness. Let $u, u' \in L^\infty(0,T;H^m)$ be solutions to (1.1)–(1.2) with $u(0) = u'(0)$ and let $w = u - u'$. Then w is a solution to

$$
\partial_t w - i\Delta w + \sum_{j=1}^N b_{1j}(t,x)\partial_j w + \sum_{j=1}^N b_{2j}(t,x)\partial_j\bar{w} + c_1(t,x)w + c_2(t,x)w = 0, \tag{4.27}
$$

$$w(0,x) = 0, \tag{4.28}$$

where

$$\begin{aligned} b_{1j}(t,x) &= -\int_0^1 \frac{\partial F}{\partial q_j}\Big(\theta u + (1-\theta)u', \theta\nabla u + (1-\theta)\nabla u'\Big)d\theta, \\ b_{2j}(t,x) &= -\int_0^1 \frac{\partial F}{\partial \bar{q}_j}\Big(\theta u + (1-\theta)u', \theta\nabla u + (1-\theta)\nabla u'\Big)d\theta, \\ c_1(t,x) &= -\int_0^1 \frac{\partial F}{\partial u}\Big(\theta u + (1-\theta)u', \theta\nabla u + (1-\theta)\nabla u'\Big)d\theta, \\ c_2(t,x) &= -\int_0^1 \frac{\partial F}{\partial \bar{u}}\Big(\theta u + (1-\theta)u', \theta\nabla u + (1-\theta)\nabla u'\Big)d\theta. \end{aligned}$$

It is easy to see

$$\begin{aligned} |\mathrm{Im}\, b_{1j}(t,x)| &\le \phi_j(t,x_j;u,u'), \\ \phi_j(t,x_j;u,u') &= M \sum_{|\alpha|\le[\frac{N-1}{2}]+3} \int_{\mathbb{R}^{N-1}} \Big(|\partial^\alpha u(t,x)|^2 + |\partial^\alpha u'(t,x)|^2\Big)d\check{x}_j \end{aligned}$$

with some large constant $M > 0$. We see (4.27)–(4.28) as an initial value problem for the 2×2 linear system of ${}^t(w,\bar{w})$. Using $\phi_j(t,x_j;u,u')$, we can show that the initial value problem for this system is L^2–well-posed by the same arguments in Proposition 2.3.2. This implies that $w(t) = 0$ for $t \in [0,T]$.

Continuity in the time variable. Finally we prove the continuity of solutions in the time variable. Let $u \in L^\infty(0,T;H^m)$ be a unique solution to (1.1)–(1.2). Then (1.1) means $\partial_t u \in L^\infty(0,T;H^{m-2})$ and therefore $u \in C^{0,1}\big([0,T];H^{m-2}\big)$. The simple interpolation yields $u \in C^{0,a}([0,T];H^{m-2a})$ for $a > 0$. Thus we have only to verify $\partial^\alpha u \in C([0,T];H^1)$ for all $\alpha \in (\mathbb{Z}_+)^N$ satisfying $|\alpha| = m-1$. We put $w_\alpha = \partial^\alpha u$, $w_{\alpha,0} = \partial^\alpha u_0$ and $g_\alpha = \partial^\alpha F - P_\alpha$. w_α is a solution to

$$\partial_t w_\alpha - i\Delta w_\alpha - \sum_{j=1}^N \frac{\partial F}{\partial q_j}\partial_j w_\alpha - \sum_{j=1}^N \frac{\partial F}{\partial \bar{q}_j}\partial_j \bar{w}_\alpha = g_\alpha,$$

$$w_\alpha(0,x) = w_{\alpha,0}(x).$$

Clearly $w_{\alpha,0} \in H^1$ and (3.3) shows $g_\alpha \in L^\infty(0,T;H^1)$. With the same arguments used for the uniqueness, we can show that the initial value problem for the 2×2 linear system of ${}^t(w_\alpha,\bar{w}_\alpha)$ is H^1–wellposed. This implies $\partial^\alpha u = w_\alpha \in C\big([0,T];H^1\big)$. This completes the proof of the continuity in the time variable and then the proof of Part (i) of Theorem 4.3.1 has been finished. ∎

Proof of Part (ii) *of Theorem 4.3.1.* We give the outline of the proof of the quadratic

case. To make the coefficient of the first order terms to be integrable on any line in $\mathbb{R}^N$, we introduce the weighted Sobolev space $H^m \cap H^{m-2,2}$. Let δ be a constant satisfying $0 < \delta \le 1$ We remark here that

$$\begin{aligned} \left|\frac{\partial F}{\partial q_j}(u(t),\nabla u(t))\right| &\le C_R|\langle D\rangle u(t,x)| \\ &\le C_R\langle x_j\rangle^{-(1+\delta)}|\langle x\rangle^{1+\delta}\langle D\rangle u(t,x)| \\ &\le C_R\langle x_j\rangle^{-(1+\delta)}\Big(\|u(t)\|_{H^{[\frac{N}{2}]+4}} + \|u(t)\|_{H^{[\frac{N}{2}]+2,2}}\Big) \\ &\le C_R\langle x_j\rangle^{-(1+\delta)} \end{aligned}$$

for $t \in [0,T]$ and for

$$u \in C\Big([0,T];H^m \cap H^{m-2,2}\Big) \quad \text{satisfying} \quad \max_{t\in[0,T]}\Big(\|u(t)\|_{H^{[\frac{N}{2}]+4}} + \|u(t)\|_{H^{[\frac{N}{2}]+2,2}}\Big) \le R.$$

We intorduce the same notations in the previous part except for ϕ_j^ε $(j = 1,\cdots,N)$, which are here defined by

$$\phi_j^\varepsilon(t,s) = M\langle s\rangle^{-(1+\delta)}$$

with some positive constant M. Then, we estimate the following quantity

$$\begin{aligned} N_4^m(u^\varepsilon(t)) &= \|u^\varepsilon(t)\|_{H^{m-1}} + \|u^\varepsilon(t)\|_{H^{m-3,2}} \\ &+ \sum_{|\alpha|=m} \|K^\varepsilon(t)\Lambda^\varepsilon(t)v_\alpha(t)\| \\ &+ \sum_{\substack{|\alpha|=m-2 \\ |\beta|=2}} \|K^\varepsilon(t)\Lambda^\varepsilon(t)v_{\alpha\beta}(t)\|, \end{aligned}$$

where

$$v_{\alpha\beta} = {}^t(x^\beta\partial^\alpha u, \overline{x^\beta\partial^\alpha u}), \quad v_\alpha = v_{\alpha 0}, \quad \alpha,\beta \in (\mathbb{Z}_+)^N.$$

Similarly, we can prove the Part (ii) of Theorem 4.3.1. We omit here the detail. ∎

Remark 4.3.2 Remark 4.2.2 holds for the case of general spatial dimensions.

Chapter 5

Global existence for semilinear Schrödinger equations

5.1 Introduction to global existence theorems

This chapter is devoted to studying the global existence theorems. In other words, we investigate when the time-local solution studied in the previous chapter can be extended globally. In general there are two types of existence theorems of the time-global solutions to nonlinear evolution equations. One seems to be the straightforward conversion of such a theorem for ordinary differential equations to that of partial differential equations, another can be seen as the stability theory of small perturbations of steady solution which is often 0-solution. The former follows from some conservation laws or somethig like them, which imply the *a priori* estimate of solutions in certain functional space. In the theory of ordinary differential equations, the global existence theorems hold when these equations and systems are Hamiltonian systems or dissipative systems. In these cases, the theorems are independent of the size of initial data. In the same way, one can obtain the *a priori* estimate of solutions when the partial differential equations are like Hamiltonian systems or dissipative systems. Such technique is usually available only to such partial differential equations. Almost of them are very simple. More complex equations or simple equations without usuful consevation laws often have a solutions blowing up in finite time. In such a case, we usually consider small perturbations of steady solution. This corresponds to the stability of constant solution of ordinary differential equations, in which the conparison theorem for single equations or the invariat region for systems play essential roles. The straightforward extension of this technique is available to getting the *a priori* estimate of solutions to some semilinear parabolic equations and systems (see [8]). For other equations and systems, however, such thechnique does not work well. For various nonlinear evolution equations which can be seen as perturbations of simple linear equations, one makes strong use of the time-decay of the fundamental solution to linear equations to get the *a priori* estimate of solutions (see e.g., [32] and [43]). With the help of such time-decay, one can obtain some invariant region in the sense of norm. In the present chapter we study the global existence of small solutions to (1.1)–(1.2) by such method. Because (1.1) has the steady solution of $u(t,x) \equiv 0$, our anaysis can be called the analysis of the stability of 0-solution.

Before presenting our results, we introduce the history of the studies on the global existence of the initial value problem for semilinear Schrödinger equations. Up to recently, most of the studies on this subject were concerned with the initial value problem (4.1)–(4.2). The nonlinear term has special structure (so-called gauge invariance) and does not contain ∇u. Then (4.1) has conservation laws

$$\begin{aligned} \int_{\mathbb{R}^N} |u(t)|^2 dx &= \int_{\mathbb{R}^N} |u_0|^2 dx, \\ E = \int_{\mathbb{R}^N} \left\{ |\nabla u(t)|^2 + G\big(|u(t)|^2\big) \right\} dx &= \int_{\mathbb{R}^N} \left\{ |\nabla u_0|^2 + G\big(|u_0|^2\big) \right\} dx, \end{aligned}$$

where

$$f(u) = -ig\big(|u|^2\big)u, \qquad G(s) = \int_0^s g(s')ds'.$$

Now we assume

$$\left|\frac{\partial f}{\partial u}(u)\right| + \left|\frac{\partial f}{\partial \bar{u}}(u)\right| \le C(1+|u|^{\rho-1}) \quad \text{for some} \quad \rho < \begin{cases} 1+\frac{4}{N-2} & (N \ge 3), \\ +\infty & (N=1,2), \end{cases} \tag{5.1}$$

$$G\big(|u|^2\big) \ge -C|u|^2 - C|u|^{q+1} \quad \text{for some} \quad q < 1 + \frac{4}{N}. \tag{5.2}$$

(5.1) ensure the existence of time-local solution $u \in C([0,T];H^1)$ for any $u_0 \in H^1$. For a power $f(u) = i|u|^{\rho-1}u$ (i.e. $g(s) = -|s|^{\rho-1}$), (5.2) requires $\rho < 1+4/N$. Simple calculation yields

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u(t)|^2 dx &= E - \int_{\mathbb{R}^N} G\big(|u(t)|^2\big) dx &\quad (5.3) \\ &\le E + C\int_{\mathbb{R}^N} |u(t)|^2 dx + C\int_{\mathbb{R}^N} |u(t)|^{q+1} dx. &\quad (5.4) \end{aligned}$$

Using the Gagliardo–Nirenberg inequality (3.1) with $r_0 = 2$, $r_1 = q+1$, $r_2 = 2$, $j_1 = 0$ and $j_2 = 1$, the last term is estimated by

$$\int_{\mathbb{R}^N} |u(t)|^{q+1} dx \le C\left(\int_{\mathbb{R}^N} |u(t)|^2 dx\right)^{[2(q+1)-N(q-1)]/4} \left(\int_{\mathbb{R}^N} |\nabla u(t)|^2 dx\right)^{N(q-1)/4}.$$

Then we have

$$\|\nabla u(t)\|_{L^2}^2 \le E + C\|u_0\|_{L^2}^2 + C\|u_0\|_{L^2}^{[2(q+1)-N(q-1)]/2} \|\nabla u(t)\|_{L^2}^{N(q-1)/2}.$$

The function $h(s) = Cs^\gamma + C - s^2$ ($s \ge 0$, $\gamma = N(q-1)/2 < 2$) is nonnegative if and only if $s \in [0, s_0]$ for some $s_0 > 0$. This implies that $\|\nabla u(t)\| \le s_0$. Then we get the *a priori* estimate and we proved the existence of time-global solution $u \in C([0,\infty);H^1)$ for any $u_0 \in H^1$. For the detail, See T. Kato's paper [29] or W. A. Strauss' lecture note [45, Chaper 3].

On the other hand, if the *a priori* estimate cannot follow from the conservation laws, then, in general, the time-local solution cannot be extended nessecerily in time. Such results are often called "blow-up thorems". This does not mean the "actual blow-up". To illustrate this fact, we consider the initial value problem

$$\begin{aligned} \partial_t u - i\partial^2 u &= i|u|^4 u \quad \text{in} \quad (0,T) \times \mathbb{R}, &\quad (5.5) \\ u(0,x) &= u_0(x) \qquad \text{in} \quad \mathbb{R}. &\quad (5.6) \end{aligned}$$

The following argument is basically due to R. Glassey's work [18], in which he proved that the time-local smooth solution to (4.1)–(4.2) could not be extended globally provided that the energy E was negative and

$$g\big(|u|^2\big)|u|^2 \ge \left(1 + \frac{2}{N}\right) G\big(|u|^2\big). \tag{5.7}$$

For a power $f(u) = i|u|^{\rho-1}u$, (5.7) requires $\rho \ge 1 + 4/N$. His idea is now the standard method to prove "blow up" theorems for some semilinear Schrödinger equations and some systems. The energy of (5.5) is

$$E = \int_{\mathbb{R}} \left\{ |\partial u(t)|^2 - \frac{1}{3}|u(t)|^6 \right\} dx.$$

Let $J' = J'(t)$ be an operator defined by $J'u = xu + it\partial u$. It is easy to see $[J', \partial_t - i\partial^2] = 0$. Simple calculation implies that a smooth solution to (5.5)–(5.6) satisies the pseudo-conformal identity

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \left\{ |J'u(t)|^2 - \frac{4}{3}t^2|u(t)|^6 \right\} dx \\ &= \frac{d}{dt}\left[\int_{\mathbb{R}} \left\{ x^2|u(t)|^2 + 4tx\mathrm{Im}\,\big(u(t)\partial u(t)\big) \right\} dx + 4t^2 E\right] = 0, \end{aligned}$$

and the dilation identity

$$\frac{d}{dt}\int_{\mathbb{R}} x\mathrm{Im}\,\big(u(t)\partial u(t)\big) dx = -2E.$$

The above identities imply

$$\begin{aligned} &\frac{d^2}{dt^2}\int_{\mathbb{R}} x^2|u(t)|^2 dx \\ &= -\frac{d^2}{dt^2}\left\{ 4t\int_{\mathbb{R}} x\mathrm{Im}\,\big(u(t)\partial u(t)\big) dx + 4t^2 E \right\} \\ &= 8E. \end{aligned} \tag{5.8}$$

We choose the initial data $u_0(x) = e^{i\theta}e^{-\varepsilon x^2}$ with some $\theta \in \mathbb{R}$. Then we have

$$\begin{aligned} E &= \int_{\mathbb{R}} \left(4\varepsilon^2 x^2 e^{-2\varepsilon x^2} - \frac{1}{3}e^{-6\varepsilon x^2}\right) dx \\ &= \int_{\mathbb{R}} \left\{4\varepsilon\left(\varepsilon x^2 e^{-\varepsilon x^2}\right) - \frac{1}{3\sqrt{6}}\right\} e^{-\varepsilon x^2} dx \\ &\leq \int_{\mathbb{R}} \left(\frac{4\varepsilon}{e} - \frac{1}{3\sqrt{6}}\right) e^{-\varepsilon x^2} dx < 0 \end{aligned}$$

provided that $\varepsilon < e/12\sqrt{6}$. Then, (5.8) implies that there exists a finite time $T_0 > 0$ such that

$$\int_{\mathbb{R}} x^2 |u(t)|^2 dx = 4Et^2 + c_1 t + c_2 < 0 \quad \text{for} \quad t > T_0,$$

which is impossible. This shows that some smooth solution to (5.5)–(5.6) cannot exist globally in time.

The above arguments assert that for a power $f(u) = -i|u|^{\rho-1}u$, the global existence theorem holds if and only if $\rho < 1+4/N$. In this case, many authors studied the scattering theory for (4.1)–(4.2). See [45, Chapter5] and its references.

Next we introduce the history of studies on the case that the nonlinear term contains ∇u. There are few works on this case because of the difficulty of the loss of derivatives. We cannot expect global existence and blow-up theorems like the above ones, because the nonlinear term $F(u, \nabla u)$ is far more complicated than $f(u)$. We cannot usually get conservation laws. In general, this is ture in the studies on various complicated nonlinear evolution equations. It seems to be natural to study small perturbation of 0-solution to complicated nonlinear evolution equations (see e.g., [32], [43], [45], [26] and their references). The basic idea of these studies is as follows. First one sees the original nonlinear equations as the perturbation of simple linear equations, whose fundamental solutions often give the time-decay. Secondly one gets the invariant region near 0-solution in the sense of norm, by using the time-decay and the smallness of the initial data. This gives the *a priori* estimate and one can prove the global existence theorems.

In our subject, S. Klainerman and G. Ponce ([32]), and J. Shatah ([43]) first studied the global existence of small amplitude solutions to (1.1)–(1.2) whose nonlinear term $F(u, q)$ satisfies

$$\operatorname{Im} \frac{\partial F}{\partial q_j}(u, q) = 0 \quad \text{for} \quad (u, q) \in \mathbb{C} \times \mathbb{C}^N, \ j = 1, \cdots, N. \tag{5.9}$$

They made strong use of the L^p–L^q estimate of $e^{it\Delta}$. As we saw in Chapter 3, the higher the spatial dimension N is, the larger the time-decay is. Moreover, the larger the power of nonlinerity ρ is, the better we make use of the time-decay. In general, the global existence of small amplitude solution to nonlinear evolution equations depends on the size of the spatial dimension N and the power of nonlinearity ρ. The sufficient condition of such

theorems is often given as the form of the lower bounds of some relationship between N and ρ.

Under the condition (5.9), the classical energy estimates are valid and then they did not have to take care of the loss of derivatives. They proved

Theorem 5.1.1 (S. Klainerman and G. Ponce ([32]), J. Shatah ([43])) *Assume* (5.9) *and*

$$\frac{N}{2}\frac{(\rho-1)^2}{\rho} > 1. \tag{5.10}$$

Let m be an integer $\geq m_0 = [N/2]+2$. Then there exists $\delta > 0$ such that if $u_0 \in H^m \cap W^{m,p}$ $(p = 2\rho/(2\rho-1))$ and u_0 satisfies

$$\|u_0\|_{H^{m_0}} + \|u_0\|_{W^{m_0,p}} \leq \delta,$$

then the initial value problem (1.1)–(1.2) *admits a unique solution $u \in C([0,\infty); H^m)$. Moreover, we have*

$$\begin{aligned} \|u(t)\|_{W^{,\infty}} &\leq C(1+t)^{-N(\rho-1)/2\rho}, \\ \|u(t)\|_{W^{,2\rho}} &\leq C(1+t)^{-N(\rho-1)/2\rho}, \\ \|u(t)\|_{H^m} &\leq C, \end{aligned}$$

for $t \in [0,\infty)$.

We remark here that in [32] or [43] they studied global existence of small amplitude solutions to various nonlinear evolution equations by the same idea. Their results are now regarded as the standard ones or the criterion on these subjects.

Secondly A. Soyeur ([44]) studied the global existence of small amplitude solutions to the initial value problem for the Ishimori equation (4.4). As we mentioned in Section 4.1, he succeeded in resolving the loss of derivatives by the gauge transformation. Combining this idea and the L^p–L^q method, he proved the global existence of small amplitude solutions to the Ishimori equation. One can easily extend his results and prove

Theorem 5.1.2 *The condition* (5.9) *in Theorem 5.1.1 can be replaced by the condition* (2.10)–(2.11).

As we studied in Section 4.2, the gauge transformation is always available when the spatial dimension is one. Recently, in view of this fact, S. Katayama and Y. Tsutsumi ([27]) proved

Theorem 5.1.3 (S. Katayama and Y. Tsutsumi ([27])) *Let $N = 1$ and let m be an integer ≥ 6. Assume that the nonlinear term satisfies*

(1) (Gauge Invariance) $F(e^{i\theta}u, e^{i\theta}q) = e^{i\theta}F(u, q)$ *for any $(u, q) \in \mathbb{C} \times \mathbb{C}$ and $\theta \in \mathbb{R}$.*

(2) *The "null gauge condition of order 3" holds for the cubic term of the Taylor expansion*

near $(u,q)=0$ (*see* [49]).
Then, there exists a constant $\delta>0$ *such that if*

$$u_0\in\bigcap_{j=0}^{5}H^{m-j,j}\quad\text{and}\quad\sum_{j=0}^{5}\|u_0\|_{H^{5-j,j}}\le\delta,$$

then the initial value problem (1.1)–(1.2) *admits a unique solution*

$$u\in\bigcap_{j=0}^{5}C\big([0,\infty);H^{m-j,j}\big).$$

We remark here that the conditions (1) and (2) require that $F(u,\partial u)$ is written by

$$\begin{aligned}F(u,q)&=\partial(|u|^2)(\alpha u+\beta\partial u)+F_5(u,\partial u)\\&=\frac{1}{2i(1+t)}(Ju\bar u-u\overline{Ju})(\alpha u+\beta\partial u)+F_5(u,\partial u),\end{aligned}\tag{5.11}$$

where $\alpha,\beta\in\mathbb{C}$ and $F_5(u,q)=O(|u|^5+|q|^5)$ near $(u,q)=0$. [27] is the first paper which investigated the global existence theorem for general semilinear Schrödinger equations. In view of (5.10), the cubic term of $F(u,\partial u)$ causes the lack of time-decay when $N=1$. Then it prevent one from proving the global existence theorem. In ([27]) they resolve this difficulty by the extra time-decay gained in (5.11).

In this chapter we present the global existence of small solutions to (1.1)–(1.2) without the restrictions on the structure of nonlinearity (like (2.10)–(2.11) in Theorem 5.1.2) and on the spatial dimension (like $N=1$ in Theorem 5.1.3). Our strategy consists of the local existence theorems studied on Chpater 4 and the *a priori* estimates obtained by the time-decay estimates developed in Chapter 3. Depending on the spatial dimension and the power of the nonlinearity, our results devide in three parts. That is

Theorem 5.1.4 (Global existence) (i) *We assume* $N\ge3$ *and* $\rho\ge3$. *Let* m_3 *be a sufficiently large integer. Then there exists a small constant* $\delta_3>0$ *such that for any*

$$u_0\in\bigcap_{j=0}^{2}H^{m-2j,j}\quad(m\ge m_3+2)\quad\text{satisfying}\quad\sum_{j=0}^{2}\|u_0\|_{H^{m_3-2j,j}}\le\delta_3,$$

the inital value problem (1.1)–(1.2) *possesses a unique solution*

$$u\in\bigcap_{j=0}^{2}C([0,\infty);H^{m-2j,j}).$$

(ii) *We assume that* $N=2$, $\rho\ge3$ *and*

$$F_3(e^{i\theta}u,e^{i\theta}q)=e^{i\theta}F_3(u,q)\quad\text{for}\quad(u,q)\in\mathbb{C}\times\mathbb{C}^N,\ \theta\in\mathbb{R}\tag{5.12}$$

where $F_3(u,q)$ *is a homogeneous cubic part of* $F(u,q)$ *near* $(u,q)=0$. *Let* m_4 *be a sufficiently large integer. Then there exists a small constant* $\delta_4>0$ *such that for any*

$$u_0\in\bigcap_{j=0}^{1}H^{m-2j,j}\quad(m\ge m_4+2)\quad\text{satisfying}\quad\sum_{j=0}^{1}\|u_0\|_{H^{m_4-2j,j}}\le\delta_4,$$

the inital value problem (1.1)–(1.2) *possesses a unique solution*

$$u\in\bigcap_{j=0}^{1}C([0,\infty);H^{m-2j,j}).$$

(iii) *We assume* $N\ge13$ *and* $\rho=2$. *Let* m_5 *be a sufficiently large integer. Then there exists a small constant* $\delta_5>0$ *such that for any*

$$u_0\in\bigcap_{j=0}^{2}H^{m-2j,j}\quad(m\ge m_5+2)\quad\text{satisfying}\quad\sum_{j=0}^{2}\|u_0\|_{H^{m_5-2j,j}}\le\delta_5,$$

the inital value problem (1.1)–(1.2) *possesses a unique solution*

$$u\in\bigcap_{j=0}^{2}C([0,\infty);H^{m-2j,j}).$$

Remark 5.1.1 Since our analysis is based on the symbolic calculus of psuedo-differential operators, it is very troublesome to determine the minimum of m_3, m_4 and m_5.

Remark 5.1.2 (5.10) yields

$$\rho\ge\begin{cases}2&(N\ge5),\\3&(N\ge2),\\4&(N\ge1).\end{cases}$$

Our condition on the relationship between the spatial dimension N and the power of nonlinearity ρ is somewhat stronger (5.10). This is basically due to the loss of time-decay associated to our transformation. The condition on the case of the cubic nonlinerity in Theorem 5.1.4 is slightly stronger than that of (5.10). On the other hand, however, that of quadratic case is far stronger than (5.10).

We prove (i), (ii), (iii) in Sections 5.2, 5.3, 5.4 respectively.

5.2 Proof of Part (i)

In this section we prove Part (i) of Theorem 5.1.4. In view of Part (i) of Theorem 4.3.1, we have only to obtain the *a priori* estimate on $\|u(t)\|_{H^{m_1}}$. Let l be the same integer as in Section 2.3. We take m_3 as $m_3 = [(N+1)/2] + l + 5 \geq m_1 + 1$. Then it is sufficient to get the *a priori* estimate on $\|u(t)\|_{H^{m_3}}$. Since we assume $u_0 \in H^{m_3+2}$, Theorem 4.3.1 and Lemma 2.3.4 ensure the validity of the energy estimates in H^{m_2}.

Let u be a solution to (1.1)–(1.2) satisfying

$$u \in \bigcap_{j=0}^{2} C([0,T]; H^{m-2j,j}), \quad \sup_{t\in[0,T]} \|u(t)\|_{W^{1,\infty}} \leq R_0$$

with some $T, R_0 > 0$ and some $m \in \mathbb{N} \geq m_3 + 2$. The local existence in this weighted Sobolev space is proved by the same method proving Theorem 4.3.1 or by [3]. To carry out the energy estimates, we introduce some notations

$$\begin{aligned}
b^1_{11j}(t,x) &= -\frac{\partial F}{\partial q_j}(u,\nabla u),\\
b^1_{12j}(t,x) &= -\frac{\partial F}{\partial \bar{q}_j}(u,\nabla u),\\
b^1_{21j}(t,x) &= \overline{\frac{\partial F}{\partial \bar{q}_j}(u,\nabla u)},\\
b^1_{22j}(t,x) &= \overline{\frac{\partial F}{\partial q_j}(u,\nabla u)},\\
b^2_{11j}(t,x) &= -G_j(u,\nabla u),\\
b^2_{12j}(t,x) &= -G'_j(u,\nabla u),\\
b^2_{21j}(t,x) &= \overline{G'_j(u,\nabla u)},\\
b^2_{22j}(t,x) &= \overline{G_j(u,\nabla u)}, \qquad j = 1,\cdots,N,
\end{aligned}$$

$$b^n(t,x,\xi) = \sum_{j=1}^{N} \begin{bmatrix} b^n_{11j}(t,x) & b^n_{12j}(t,x) \\ b^n_{21j}(t,x) & b^n_{22j}(t,x) \end{bmatrix} \xi_j, \quad n = 1,2.$$

In the same way as in Section 2.3, we define $B_{b^n}(t), n = 1,2$ and we put $B_b(t) = B_{b^1}(t) + B_{b^2}(t)$. By the Gagliardo–Nirenberg inequality (the interpolation between $W^{1,\infty}$ and H^{m_2-1} for $W^{l+3,\infty}$) and (3.19), we have

$$B_b(t) \leq C_{R_0}(1+t)^{-(1+\varepsilon)} X^N_{m_2-1}(t)^2 \quad \text{for} \quad t \in [0,T], \tag{5.13}$$

$$\varepsilon = \frac{7}{4}\frac{m_3 - N/2 - 2l/7 - 4}{m_3 - N/2 - 4} > 0.$$

Using (3.11), in the same way as Section 4.3, we have

$$|\mathrm{Im}\, b^n_{hhj}(t,x)| \leq \phi_j(t,x_j), \quad \text{for} \quad (t,x) \in [0,T]\times\mathbb{R}^N, \quad n,h = 1,2, \ j = 1,\cdots,N, \tag{5.14}$$

$$\phi_j(t,x_j) = M(1+t)^{-3/2} \sum_{\substack{|\alpha|\leq[\frac{N+1}{2}]+1 \\ |\beta|\leq 1}} \int_{\mathbb{R}^{N-1}} |\partial^\alpha J^\beta u(t,x)|^2 d\check{x}_j, \quad j = 1,\cdots,N,$$

where $M = M(R_0,F) > 0$ is a sufficiently large constant. We introduce the pseudo-differential operators defined as follows.

$$K(t) = k(t,x,D), \quad k(t,x,\xi) = \begin{bmatrix} e^{-p(t,x,\xi)} & 0 \\ 0 & e^{p(t,x,\xi)} \end{bmatrix},$$

$$p(t,x,\xi) = -\sum_{j=1}^{N} \int_0^{x_j} \phi_j(t,s)ds\,\xi_j\langle\xi_j\rangle^{-1},$$

$$\Lambda^n(t) = I + \tilde{\Lambda}^n(t), \quad \tilde{\Lambda}^n(t) = \tilde{\Lambda}^n(t,x,D),$$

$$\tilde{\Lambda}^n(t,x,\xi) = \frac{1}{2}\sum_{j=1}^{N} \begin{bmatrix} 0 & b^n_{12j}(t,x) \\ -b^n_{21j}(t,x) & 0 \end{bmatrix} \xi_j\langle\xi\rangle^{-2}, \quad n = 1,2.$$

We also define $B_K(t)$, $B^0_\phi(t)$, $B^1_\phi(t)$ and $B^\infty_\phi(t)$. Simple calculation yields

$$B_K(t), B^0_\phi(t) \leq A(X^N_{m_2-1}(t)), \tag{5.15}$$

$$B^1_\phi(t), B^\infty_\phi(t) \leq C_{R_0}(1+t)^{-3/2} X^N_{m_2-1}(t)^2, \tag{5.16}$$

where $A(\cdot)$ is a non-decreasing function on $[0,\infty)$ depending on R_0. We put

$$\begin{aligned}
v_\alpha &= {}^t(\partial^\alpha u, \partial^\alpha \bar{u}), \quad v_{\alpha,0} = {}^t(\partial^\alpha u_0, \partial^\alpha \overline{u_0}),\\
f_\alpha &= {}^t(f_{\alpha,1}, f_{\alpha,2}), \quad f_{\alpha,1} = \overline{f_{\alpha,2}}\\
f_{\alpha,1} &= \partial^\alpha F - P_\alpha, \qquad \text{for} \quad |\alpha| = m_3,
\end{aligned}$$

and

$$\begin{aligned}
v_{\alpha\beta} &= {}^t(\partial^\alpha J^\beta u, \overline{\partial^\alpha J^\beta u}), \quad v_{\alpha\beta,0} = v_{\alpha\beta}(0) = {}^t\Big(\partial^\alpha(x^\beta u_0), \overline{\partial^\alpha(x^\beta u_0)}\Big),\\
f_{\alpha\beta} &= {}^t(f_{\alpha\beta,1}, f_{\alpha\beta,2}), \quad f_{\alpha\beta,1} = \overline{f_{\alpha\beta,2}}\\
f_{\alpha\beta,1} &= \partial^\alpha J^\beta F - P_{\alpha\beta}, \qquad \text{for} \quad |\alpha + 2\beta| = m_3, \ |\beta| = 1,2.
\end{aligned}$$

We evaluate $Y^N_{m_3}(t)$ which is defined by

$$\begin{aligned}
Y^N_{m_2}(t) &= X^N_{m_2-1}(t) + \sum_{|\alpha|=m_3} \|K(t)\Lambda_1(t)v_\alpha(t)\|\\
&+ \sum_{\substack{|\alpha|=m_3-2\\|\beta|=1}} \|K(t)\Lambda_2(t)v_{\alpha\beta}(t)\| + (1+t)^{-1/4} \sum_{\substack{|\alpha|=m_3-4\\|\beta|=2}} \|K(t)\Lambda_2(t)v_{\alpha\beta}(t)\|.
\end{aligned}$$

We suppose that there exists a constant $R > 0$ such that

$$\sup_{t\in[0,T]} Y^N_{m_3}(t) \leq R.$$

In view of (2.29), (2.30), (5.13), (5.15) and (5.16), we have

$$C_R^{-1} Y_{m_3}^N(t) \le X_{m_3}^N(t) \le C_R Y_{m_3}^N(t), \quad Y_{m_3}^N(0) \le C_R \delta, \tag{5.17}$$

with some constant $C_R > 0$. Using (3.22), (3.23), (3.24) and (5.17), we get

$$\|F(t)\|_{H^{m_3-1}} + \sum_{|\alpha|=m_3} \|K(t)\Lambda^1(t) f_\alpha(t)\| \le C_R(1+t)^{-11/4} R^3, \tag{5.18}$$

$$\|JF(t)\|_{H^{m_3-3}} + \sum_{\substack{|\alpha|=m_3-2\\|\beta|=1}} \|K(t)\Lambda^2(t) f_{\alpha\beta}(t)\| \le C_R(1+t)^{-7/4} R^3, \tag{5.19}$$

$$\|J^2F(t)\|_{H^{m_3-5}} + \sum_{\substack{|\alpha|=m_3-4\\|\beta|=2}} \|K(t)\Lambda^2(t) f_{\alpha\beta}(t)\| \le C_R(1+t)^{-3/4} R^3. \tag{5.20}$$

By (5.13), (5.15) and (5.16), we have

$$\Big(B_b(t) + B_\phi^1(t) + B_\phi^\infty(t)\Big) B_{\mathrm{etc}}(t) \le C_R(1+t)^{-(1+\varepsilon)} R^2. \tag{5.21}$$

v_α and $v_{\alpha\beta}$ satisfy

$$\begin{aligned} \Big(I\partial_t + i(a(D) + b^1(t,x,D))\Big) v_\alpha &= f_\alpha, \\ \Big(I\partial_t + i(a(D) + b^2(t,x,D))\Big) v_{\alpha\beta} &= f_{\alpha\beta}, \end{aligned}$$

respectively. Using (2.47), (5.14), (5.18), (5.19), (5.20) and (5.21), we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{|\alpha|=m_3} \|K(t)\Lambda^1(t) v_\alpha(t)\| &\le C_R(1+t)^{-(1+\varepsilon)} R^3, \\ \frac{d}{dt} \sum_{\substack{|\alpha|=m_3-2\\|\beta|=1}} \|K(t)\Lambda^2(t) v_\alpha(t)\| &\le C_R(1+t)^{-(1+\varepsilon)} R^3, \\ \frac{d}{dt} \sum_{\substack{|\alpha|=m_3-4\\|\beta|=2}} \|K(t)\Lambda^2(t) v_\alpha(t)\| &\le C_R(1+t)^{-3/4} R^3. \end{aligned}$$

The integration on $[0,t]$ implies

$$\begin{aligned} \sum_{|\alpha|=m_3} \|K(t)\Lambda^1(t) v_\alpha(t)\| + \sum_{\substack{|\alpha|=m_3-2\\|\beta|=1}} \|K(t)\Lambda^2(t) v_{\alpha\beta}(t)\| & \\ +(1+t)^{-1/4} \sum_{\substack{|\alpha|=m_3-4\\|\beta|=2}} \|K(t)\Lambda^2(t) v_{\alpha\beta}(t)\| &\le C_R(\delta + R^3). \end{aligned} \tag{5.22}$$

Noting $[J, \partial_t - i\Delta] = 0$ we have

$$(\partial_t - i\Delta)\partial^\alpha J^\beta u = \partial^\alpha J^\beta F, \quad \text{with} \quad |\alpha + 2\beta| \le m_3 - 1, \ |\beta| \le 2.$$

Making use of (5.18), (5.19) and (5.20), we obtain

$$X_{m_3-1}^N(t) \le C_R(\delta + R^3). \tag{5.23}$$

Combining (5.22) and (5.23), we get

$$\sup_{t\in[0,T]} Y_{m_3}^N(t) \le C_R(\delta + R^3).$$

Then there exist constants $R_1 > 0$ and $C_{R_1} > 0$ which are independent of $T > 0$, such that

$$\sup_{t\in[0,T]} Y_{m_3}^N(t) \le C_{R_1}(\delta + R^3), \quad \text{if} \quad R \le R_1.$$

If we choose R_1 as $C_{R_1} R_1^3 \le R_1/4$ at the beginning, then we have

$$\sup_{t\in[0,T]} Y_{m_3}^N(t) \le R/2 \quad \text{if} \quad R \le R_1$$

provided that δ is sufficiently small. This completes the proof of Part (i) of Theorem 5.1.4.

5.3 Proof of Part (ii)

In this section we prove Part (ii) of Theorem 5.1.4. The outline of its proof is basically the same as that of Part (i). Let l be the same integer as in Section 2.3. We take m_4 as $m_4 = [(3l+1)/2] + 7 \ge m_1 + 1$. We obtain the *a priori* estimate in $H^{m_4} \cap H^{l+3,1}$. Let m be an integer $\ge m_4 + 2$ and let u be a solution to (1.1)–(1.2) satisfying

$$u \in \bigcap_{j=0}^{2} C\Big([0,T]; H^{m-2j,j}\Big).$$

We define $\phi_j(t, x_j)$ by

$$\phi_j(t,x_j) = M(1+t)^{-1} \sum_{\substack{|\alpha|\le 1\\|\beta|\le 1}} \int_{\mathbb{R}} |\partial^\alpha J^\beta J^\beta u(t,x)|^2 d\check{x}_j, \quad j = 1,2,$$

with some large constant $M > 0$. We introduce $K(t)$, $\Lambda^n(t)$, $b^n(t,x,\xi)$, $n = 1,2$, $B_K(t)$, $B_b(t)$, $B_\phi^0(t)$, $B_\phi^1(t)$ and $B_\phi^\infty(t)$ in the same way as in the previous section. We put

$$\begin{aligned} Z(t) &= Z_1(t) + (1+t)^{-\varepsilon} Z_2(t), \\ Z_1(t) &= \|u(t)\|_{H^{m_4-1}} + \|Ju(t)\|_{H^{l+2}}, \\ Z_2(t) &= \sum_{|\alpha|=m_4} \|K(t)\Lambda^1(t) v_\alpha(t)\| + \sum_{\substack{|\alpha|=l+3\\|\beta|=1}} \|K(t)\Lambda^2(t) v_{\alpha\beta}(t)\|, \\ \varepsilon &= \frac{m_4 - (3l+1)/2 - 6}{8(m_4 - 2)} > 0. \end{aligned}$$

We suppose that there exists a constant $R > 0$ such that

$$\sup_{t\in[0,T]} Z(t) \le R.$$

Then we have

$$|\mathrm{Im}\, b^n_{hhj}(t,x)\| \le \phi_j(t,x_j),$$
$$\text{for} \quad (t,x) \in [0,T] \times \mathbb{R}^N, \quad n,h, = 1,2, \quad j = 1,\cdots,N, \tag{5.24}$$

$$B_K(t), B^0_\phi(t) \le C_R, \tag{5.25}$$
$$B^1_\phi(t), B^\infty_\phi(t) \le C_R(1+t)^{-1}R^2, \tag{5.26}$$
$$B_b(t) \le C_R(1+t)^{-(1+2\varepsilon)}R^2, \tag{5.27}$$
$$\big(B_b(t) + B^1_\phi(t) + B^\infty_\phi(t)\big)B_{\mathrm{etc}}(t) \le C_R(1+t)^{-1}R^2, \tag{5.28}$$

$$\|F(t)\|_{H^{m_4-1}} + \sum_{|\alpha|=m_4} \|K(t)\Lambda^1(t)f_\alpha(t)\|$$
$$+\|JF(t)\|_{H^{l+2}} + \sum_{\substack{|\alpha|=l+3\\|\beta|=1}} \|K(t)\Lambda^2(t)f_{\alpha\beta}(t)\| \le C_R(1+t)^{-(1+\varepsilon)}R^3. \tag{5.29}$$

Here we used (3.26), (3.27) and (3.36) to get (5.27) and (5.28). We here show the estimate of $B^1_\phi(t)$ actually, because its bound is delicate.

$$(1+t)\left|\int_0^{x_j} \partial_t\phi_j(t,y_j)dy_j\right|$$
$$= \left|-M(1+t)^{-1}\sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} |\partial^\alpha J^\beta u(t,y_j,\check{x}_j)|^2 d\check{x}_j dy_j\right.$$
$$\left. + \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \partial_t|\partial^\alpha J^\beta u(t,y_j,\check{x}_j)|^2 d\check{x}_j dy_j\right|$$
$$\le C(1+t)^{-1}Z_1(t) + \left|\sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \partial_t|\partial^\alpha J^\beta u(t,y_j,\check{x}_j)|^2 d\check{x}_j dy_j\right|$$

We have only to show that the last term is $O(1)$ for any $t \ge 0$. In the same way as (4.8), we have

$$\sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \partial_t|\partial^\alpha J^\beta u(t,y_j,\check{x}_j)|^2 d\check{x}_j dy_j$$

$$= \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \left(\partial_t\partial^\alpha J^\beta u\overline{\partial^\alpha J^\beta u} + \partial^\alpha J^\beta u\partial_t\overline{\partial^\alpha J^\beta u}\right)(t,y_j,\check{x}_j)d\check{x}_j dy_j$$
$$= \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \left(i\Delta\partial^\alpha J^\beta u\overline{\partial^\alpha J^\beta u} - \partial^\alpha J^\beta u i\Delta\overline{\partial^\alpha J^\beta u}\right)(t,y_j,\check{x}_j)d\check{x}_j dy_j$$
$$+ \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \left(\partial^\alpha J^\beta F(u,\nabla u)\overline{\partial^\alpha J^\beta u} + \partial^\alpha J^\beta u\overline{\partial^\alpha J^\beta F(u,\nabla u)}\right)(t,y_j,\check{x}_j)d\check{x}_j dy_j$$
$$= \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_{\mathbb{R}^{N-1}} \left(i\partial_j\partial^\alpha J^\beta u\overline{\partial^\alpha J^\beta u} - \partial^\alpha J^\beta u i\partial_j\overline{\partial^\alpha J^\beta u}\right)(t,x_j,\check{x}_j)d\check{x}_j$$
$$+ \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_{\mathbb{R}^{N-1}} \left(i\partial_j\partial^\alpha J^\beta u\overline{\partial^\alpha J^\beta u} - \partial^\alpha J^\beta u i\partial_j\overline{\partial^\alpha J^\beta u}\right)(t,0,\check{x}_j)d\check{x}_j$$
$$+ \sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_0^{x_j}\int_{\mathbb{R}^{N-1}} \left(\partial^\alpha J^\beta F(u,\nabla u)\overline{\partial^\alpha J^\beta u} + \partial^\alpha J^\beta u\overline{\partial^\alpha J^\beta F(u,\nabla u)}\right)(t,y_j,\check{x}_j)d\check{x}_j dy_j$$
$$\le C\sup_{x_j\in\mathbb{R}}\sum_{\substack{|\alpha|\le1\\|\beta|\le1}} |\partial^\alpha J^\beta u(t,x)|^2 d\check{x}_j$$
$$+ 2\sum_{\substack{|\alpha|\le1\\|\beta|\le1}}\int_{\mathbb{R}^N} \left|\partial^\alpha J^\beta F(u,\nabla u)\overline{\partial^\alpha J^\beta u}\right|(t,x)dx.$$

We remark here that

$$\partial^\alpha J^\beta F(u,\nabla u)\overline{\partial^\alpha J^\beta u} = \text{forth degree term of } (u,\partial u, Ju, \partial Ju)$$
$$+ (1+t)\text{fifth degree term of } u, \partial u, Ju, \partial Ju).$$

By (3.20), the last term is estimated by C_R. $B^1_\phi(t)$ does not cause the loss of time-decay. Unless F_3 satisfies the gauge invariance (5.12), $B^1_\phi(t)$ gives the loss of time-decay, which prevent us from proving the global existence theorems. This property was first pointed out by S. Katayama and Y. Tsutsumi ([27]).

v_α and $v_{\alpha\beta}$ satisfy

$$\big(I\partial_t + i(a(D) + b^1(t,x,D))\big)v_\alpha = f_\alpha,$$
$$\big(I\partial_t + i(a(D) + b^2(t,x,D))\big)v_{\alpha\beta} = f_{\alpha\beta},$$

respectively. Using (2.47) (5.24), (5.25), (5.26), (5.27), (5.28) and (5.29), we obtain

$$\frac{d}{dt}Z_2(t) \le C_R(1+t)^{-1+\varepsilon}R^3.$$

The integration on $[0,t]$ yields

$$(1+t)^{-\varepsilon}Z_2(t) \le C_R(\delta + R^3). \tag{5.30}$$

In the same way as (5.23) we have

$$Z_1(t) \le C_R(\delta + R^3). \tag{5.31}$$

Combining (5.30) and (5.31), we get

$$\sup_{t\in[0,T]} Z(t) \le C_R(\delta + R^3).$$

Then there exist constants $R_1 > 0$ and $C_{R_1} > 0$ which are independent of $T > 0$, such that

$$\sup_{t\in[0,T]} Z(t) \le C_{R_1}(\delta + R^3), \quad \text{if} \quad R \le R_1.$$

If we choose R_1 as $C_{R_1}R_1^3 \le R_1/4$ at the beginning, then we have

$$\sup_{t\in[0,T]} Z(t) \le R/2 \quad \text{if} \quad R \le R_1$$

provided that δ is sufficiently small. This completes the proof of Part (ii) of Theorem 5.1.4.

5.4 Proof of Part (iii)

Finally, in this section we prove Part (iii) of Theorem 5.1.4. According to Theorem 4.3.1 and Remark 4.3.2, we have only to get the a *priori* estimate of

$$\sum_{\substack{|\alpha\beta+2|\beta|\le m_5\\ |\beta|\le 2}} \|\partial^\alpha J^\beta u(t)\|_{L^2}$$

with $m_5 = [(N+1)/2]+l+5 \ge m_2+1$. In the previous two sections we got the time-decay estimates by using the operator J and the Gagliardo–Nirenberg inequality. In the present section, however, we employ the L^p-L^q estimate developed in Section 3.3, because the quaqdratic nonlinearity is not suited to the previous method.

Similarly, we make use of the convenient notations $b^n(t,x,\xi)$, $\Lambda^n(t) = \lambda^n(t,x,D)$ $(n = 1,2)$, $B_b(t)$, v_α, $v_{\alpha\beta}$, f_α, $f_{\alpha\beta}$ and etc. In this section we put $\phi_j(t,s) = M(1+t)^{-(1+\varepsilon)}\langle s\rangle^{-(1+\eta)}$ with some constants $M, \varepsilon, \eta > 0$. ε and η are arbitrary small constants. M is determined as follows.

Simple calculation yields

$$\begin{aligned}
&\left|\mathrm{Im}\,\frac{\partial F}{\partial q_k}(u(t),\nabla u(t))\right| \\
\le\ & C\Big(|u(t,x)| + |\nabla u(t,x)|\Big) \\
\le\ & C\langle x_k\rangle^{-(1+\eta)}\Big(|(1+x_k^2)^{-(1+\eta)/2}u(t,x)| + |(1+x_k^2)^{-(1+\eta)/2}\nabla u(t,x)|\Big) \\
\le\ & C\langle x_k\rangle^{-(1+\eta)}\Big\{\|u(t)\|_{W^{1,\infty}} \\
&\quad + \|u(t)\|_{W^{1,\infty}}^{(1-\eta)/2}\Big(|x_k^2u(t,x)| + |x_ku(t,x)| + |\nabla(x_k^2u(t,x))|\Big)^{(1+\eta)/2}\Big\}.
\end{aligned}$$

Noting

$$\begin{aligned}
x_ku &= J_ku - 2i(1+t)\partial_ku, \\
x_k^2u &= J_k^2u - 2i(1+t)\partial_kJ_ku + 2i(1+t)u + 4(1+t)^2\partial_k^2u,
\end{aligned}$$

we get

$$\begin{aligned}
&|x_k^2u(t,x)| + |x_ku(t,x)| + |\nabla(x_k^2u(t,x))| \\
\le\ & C\Big(\|J^2u(t)\|_{W^{1,\infty}} + (1+t)\|Ju(t)\|_{W^{2,\infty}} + (1+t)^2\|u(t)\|_{W^{3,\infty}}\Big) \\
\le\ & C(-N/4+2)\sum_{\substack{|\alpha|+2|\beta|\le m_5-[N/2]-2\\ |\beta|\le 2}} (1+t)^{N/4-|\beta|}\|\partial^\alpha J^\beta u(t)\|_{L^4}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\left|\mathrm{Im}\,\frac{\partial F}{\partial q_k}(u(t),\nabla u(t))\right| \\
\le\ & C\langle x_k\rangle^{-(1+\eta)}(1+t)^{-N/4+1+\eta} \\
&\times \sum_{\substack{|\alpha|+2|\beta|\le m_5-[N/2]-2\\ |\beta|\le 2}} (1+t)^{N/4-|\beta|}\|\partial^\alpha J^\beta u(t)\|_{L^4}.
\end{aligned} \tag{5.32}$$

Let R_1 be some positive constant. We choose M as $M = CR_1$.

We estimate

$$\begin{aligned}
\tilde{Y}_{m_5}^N(t) =\ & \sum_{|\alpha|=m_5} \|K(t)\Lambda^1 v_\alpha(t)\| \\
&+ \sum_{\substack{|\alpha|+2|\beta|=m_5\\ |\beta|=1,2}} \|K(t)\Lambda^2 v_{\alpha\beta}(t)\| \\
&+ \sum_{\substack{|\alpha|+2|\beta|=m_5-1\\ |\beta|\le 2}} \|\partial^\alpha J^\beta u(t)\|_{L^2} \\
&+ \sum_{\substack{|\alpha|+2|\beta|=m_5-[N/2]-2\\ |\beta|\le 2}} \|\partial^\alpha J^\beta u(t)\|_{L^4}.
\end{aligned}$$

We suppose

$$\sup_{t\in[0,T]} \tilde{Y}_{m_5}^N(t) \le R$$

with some constant $R(\le R_1)$. (5.32) implies that

$$\left|\mathrm{Im}\,\frac{\partial F}{\partial q_k}(u(t),\nabla u(t))\right| \le \phi_j(t,s) = M(1+t)^{-(1+\varepsilon)}\langle s\rangle^{-(1+\eta)}$$

for $t \in [0,T]$. It is easy to see

$$\begin{aligned} B_K(t) &\leq C_{R_1}, && (5.33)\\ B_\phi^0(t), B_\phi^1(t), B_\phi^\infty(t), B_b(t) &\leq C_{R_1}(1+t)^{-N/4+2}R && (5.34)\end{aligned}$$

In view of (2.29), (2.30), (5.33) and (5.34), we have

$$C_{R_1}^{-1}\tilde{Y}_{m_5}^N(t) \leq \tilde{X}_{m_5}^N(t) \leq C_{R_1}\tilde{Y}_{m_5}^N(t), \qquad \tilde{Y}_{m_5}^N(0) \leq C_{R_1}\delta.$$

Then, (3.43), (3.44) and (3.45) become

$$\begin{aligned} &\|F(t)\|_{W^{m_5-1,4/3}} + \|F(t)\|_{H^{m_5-1}} + \sum_{|\alpha|=m_5}\|f_\alpha(t)\|\\ \leq\ & C_{R_1}(1+t)^{-N/4}R^2, && (5.35)\\ &\|JF(t)\|_{W^{m_5-3,4/3}} + \|JF(t)\|_{H^{m_5-3}} + \sum_{\substack{|\alpha|+2|\beta|=m_5\\|\beta|=1}}\|f_{\alpha\beta}(t)\|\\ \leq\ & C_{R_1}(1+t)^{-N/4+1}R^2, && (5.36)\\ &\|J^2F(t)\|_{W^{m_5-5,4/3}} + \|J^2F(t)\|_{H^{m_5-5}} + \sum_{\substack{|\alpha|+2|\beta|=m_5\\|\beta|=2}}\|f_{\alpha\beta}(t)\|\\ \leq\ & C_{R_1}(1+t)^{-N/4+2}R^2. && (5.37)\end{aligned}$$

v_α and $v_{\alpha\beta}$ satisfy

$$\begin{aligned} \Big(I\partial_t + i(a(D)+b^1(t,x,D))\Big)v_\alpha &= f_\alpha,\\ \Big(I\partial_t + i(a(D)+b^2(t,x,D))\Big)v_{\alpha\beta} &= f_{\alpha\beta},\end{aligned}$$

respectively. Using (2.47), (5.33), (5.34), (5.35), (5.36) and (5.37), we obtain

$$\begin{aligned} \frac{d}{dt}\sum_{|\alpha|=m_5}\|K(t)\Lambda^1(t)v_\alpha(t)\| &\leq C_{R_1}(1+t)^{-(1+\varepsilon)}R^2,\\ \frac{d}{dt}\sum_{\substack{|\alpha|=m_5-2\\|\beta|=1}}\|K(t)\Lambda^2(t)v_{\alpha\beta}(t)\| &\leq C_{R_1}(1+t)^{-(1+\varepsilon)}R^2,\\ \frac{d}{dt}\sum_{\substack{|\alpha|=m_5-4\\|\beta|=2}}\|K(t)\Lambda^2(t)v_{\alpha\beta}(t)\| &\leq C_{R_1}(1+t)^{-3/4}R^2.\end{aligned}$$

The integration on $[0,t]$ implies

$$\begin{aligned} &\sum_{|\alpha|=m_5}\|K(t)\Lambda^1(t)v_\alpha(t)\| + \sum_{\substack{|\alpha|=m_5-2\\|\beta|=1}}\|K(t)\Lambda^2(t)v_{\alpha\beta}(t)\|\\ &\qquad + \sum_{\substack{|\alpha|=m_5-4\\|\beta|=2}}\|K(t)\Lambda^2(t)v_{\alpha\beta}(t)\| \leq C_R(\delta+R^3). \qquad (5.38)\end{aligned}$$

On the other hand, using

$$(\partial_t - i\Delta)\partial^\alpha J^\beta u = \partial^\alpha J^\beta F,$$

(5.35), (5.36) and (5.37), we have

$$\begin{aligned} &\sum_{\substack{|\alpha|+2|\beta|\leq m_5-1\\|\beta|\leq 2}}\|\partial^\alpha J^\beta u(t)\|_{L^2}\\ \leq\ & \delta + \sum_{\substack{|\alpha|+2|\beta|\leq m_5-1\\|\beta|\leq 2}}\int_0^t\|\partial^\alpha J^\beta F(\tau)d\tau\\ \leq\ & C_{R_1}(\delta+R^2), && (5.39)\\ &\sum_{\substack{|\alpha|+2|\beta|\leq m_5-[N/2]-2\\|\beta|\leq 2}}(1+t)^{N/4+|\beta|}\|\partial^\alpha J^\beta u(t)\|_{L^4}\\ \leq\ & C\delta + \sum_{\substack{|\alpha|+2|\beta|\leq m_5-[N/2]-2\\|\beta|\leq 2}}\int_0^t(1+t-\tau^{-N/4})(1+t)^{N/4+|\beta|}\|\partial^\alpha J^\beta F(\tau)\|_{L^{4/3}}d\tau\\ \leq\ & C_{R_1}(\delta+R^2). && (5.40)\end{aligned}$$

Combining (5.38), (5.39) and (5.39), we obtain

$$\tilde{Y}_{m_5}^N(t)C_{R_1}(\delta+R^2).$$

Similarly, we can get the *a priori* estimate provided that $\delta > 0$ is sufficiently small. This completes the proof of Theorem 5.1.4.

Chapter 6

The initial value problem for the elliptic–hyperbolic Davey–Stewartson equation

6.1 Introduction to the Davey–Stewartson equation

In this chapter we study the initial value problem for the elliptic–hyperbolic Davey–Stewartson equation of the form

$$\begin{aligned} \partial_t u - i(\partial_x^2 + \partial_y^2)u &= f(u, \partial_x u, \partial_y u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, & (6.1)\\ u(0,x,y) &= u_0(x,y) \quad \text{in} \quad \mathbb{R}^2, & (6.2)\end{aligned}$$

where $u(t,x,y)$ is $\mathbb{C}$–valued, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, and the nonlinear term $f(u)$ is defined by

$$\begin{aligned} f(u,\partial_x u,\partial_y u) &= \sum_{j=0}^{2} a_j f_j(u,\partial_x u,\partial_y u),\\ f_0(u,\partial_x u,\partial_y u) &= |u|^2 u,\\ f_1(u,\partial_x u,\partial_y u) &= \int_y^{+\infty} \partial_x\Big(|u(x,y')|^2\Big)dy' u(x,y),\\ f_2(u,\partial_x u,\partial_y u) &= \int_x^{+\infty} \partial_y\Big(|u(x',y)|^2\Big)dx' u(x,y),\end{aligned}$$

$a_0, a_1, a_2, \in \mathbb{C}$ are constants.

We use the following notations. $(\xi,\zeta)\in\mathbb{R}^2$ means the dual variable of $(x,y)\in\mathbb{R}^2$ under the Fourier transformation. $\partial_\xi = \partial/\partial\xi$ and $\partial_\zeta = \partial/\partial\zeta$. J_x and J_y are defined by

$$\begin{aligned} J_x u &= e^{ix^2/4(1+t)} 2i(1+t)\partial_x\Big(e^{-ix^2/4(1+t)}u\Big) = (x + (1+t)\partial_x)u,\\ J_y u &= e^{iy^2/4(1+t)} 2i(1+t)\partial_y\Big(e^{-iy^2/4(1+t)}u\Big) = (y + (1+t)\partial_y)u.\end{aligned}$$

$\langle D_x\rangle = (1-\partial_x^2)^{1/2}$, $\langle D_y\rangle = (1-\partial_y^2)^{1/2}$, $\langle D_x; D_y\rangle = (1-\partial_x^2-\partial_y^2)^{1/2}$, $\langle x\rangle = \sqrt{1+x^2}$, $\langle y\rangle = \sqrt{1+y^2}$, $\langle x;y\rangle = \sqrt{1+x^2+y^2}$.

$$W^{m,p} = W^{m,p}(\mathbb{R}^2) = \left\{u \in \mathcal{S}'(\mathbb{R}^2) \;\middle|\; \|u\|_{W^{m,p}} = \left(\iint_{\mathbb{R}^2} |\langle D_x; D_y\rangle^m u|^p dxdy\right)^{1/p} < +\infty\right\},$$

$L^p = W^{0,p}$, $H^m = W^{m,2}$ for $m \in \mathbb{R}$ and $1 \le p < \infty$.

$$W^{m,\infty} = W^{m,\infty}(\mathbb{R}^2) = \left\{u \in \mathcal{S}'(\mathbb{R}^2) \;\middle|\; \|u\|_{W^{m,\infty}} = \text{ess.sup}|\langle D_x; D_y\rangle^m u| < +\infty\right\},$$

$L^\infty = W^{0,\infty}$, for $m \in \mathbb{R}$.

$$H^{m,n} = H^{m,n}(\mathbb{R}^2) = \left\{u \in \mathcal{S}'(\mathbb{R}^2) \;\middle|\; \|u\|_{H^{m,n}} = \|\langle x;y\rangle^n \langle D_x; D_y\rangle^m u\|_{L^2} < +\infty\right\}$$

for $m, n \in \mathbb{R}$. $\|\cdot\|_m$ means H^m–norm. Especially $\|\cdot\|$ and $(\cdot,\cdot)$ mean L^2–norm and L^2–inner product respectively. $\mathcal{S} = \mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^2)$ denote the Schwartz class and its topological dual respectively. Let Ω be an open subset of some Euclidean space. $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the dual pair of the class of test functions of $C_0^\infty(\Omega)$ and the class of distributions on Ω. $\mathcal{B}^0(\Omega)$ is the Banach space of all bounded linear continuous functions on Ω. $\mathcal{B}^\infty(\Omega)$ is the Fréchet space of all C^∞ functions on Ω whose derivatives of any order are all bounded. Let E and F be Fréchet spaces. $\mathcal{L}(E,F)$ denotes the set of all bounded linear operators of E to F. $\mathcal{L}(E) = \mathcal{L}(E,E)$. Let (X,Y) is a dual pair of locally convex spaces X and Y. $\langle y, x\rangle$ means the operation of $y \in Y$ on $x \in X$. $C\big([0,T]; E\big)$ and $C_{\mathrm{W}}\big([0,T]; E\big)$ are the sets of all strongly and weakly continuous E–valued functions on $[0,T]$ respectively. $[s]$ means the largest integer less than or equal to $s \in \mathbb{R}$. $\mathbb{N} = \{1,2,3,\cdots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We denote the positive constants by the same letter C.

Originally the Davey–Stewartson systems are written as

$$\partial_t u - i(\delta\partial_x^2 + \partial_y^2)u = i\gamma|u|^2 u + ib(\partial_x\varphi)u, \tag{6.3}$$

$$(\partial_x^2 + c\partial_y^2)\varphi = \partial_x\big(|u|^2\big), \tag{6.4}$$

where $\delta, \gamma = \pm 1$, $b \in \mathbb{R}$ and $c \in \mathbb{R}\backslash\{0\}$. In [16] J.-M. Ghidaglia and J.-C. Saut classified (6.3)–(6.4) as elliptic–elliptic, hyperbolic–elliptic, elliptic–hyperbolic and hyperbolic–hyperbolic according to the respective sign of $(\delta, c) = (+,+)$, $(-,+)$, $(+,-)$ and $(-,-)$, and studied the initial value problem for (6.3)–(6.4).

For the cases of the elliptic–elliptic and the hyperbolic–elliptic (i.e. $c > 0$), (6.3)–(6.4) becomes

$$\partial_t u - i(\delta\partial_x^2 + \partial_y^2)u = i\gamma|u|^2 u + ib\Big(R_c(|u|^2)\Big)u, \tag{6.5}$$

where R_c is a singular integral operator whose symbol is $\xi^2/(\xi^2 + c\zeta^2)$. Since R_c is a bounded linear operator from $L^p(\mathbb{R}^2;\mathbb{R})$ to $L^p(\mathbb{R}^2;\mathbb{R})$ for any $1 < p < +\infty$, (6.5) is similar to

$$\partial_t u - i(\delta\partial_x^2 + \partial_y^2)u = i\gamma'|u|^2 u, \qquad \gamma' \in \mathbb{R}\backslash\{0\}.$$

Then J.-M. Ghidaglia and J.-C. Saut ([16]) obtained the complete results on the local existence, the global existence and the blow–up of the initial value problem for (6.5) under the condition $\delta = +1$. If $\delta = -1$, the global existence is an open problem even for the small initial data.

On the other hand, in the cases of the elliptic–hyperbolic and the hyperbolic–hyperbolic (i.e. $c < 0$), one assume the radiation condition

$$\varphi(t,x,y) \longrightarrow 0 \quad \text{as} \quad x+y, x-y \longrightarrow +\infty \tag{6.6}$$

in order that (6.4) is solvable. Here we put $c = -1$ for simplicity. With the transformation $x := (x+y)/2$, $y := (x-y)/2$, (6.3)–(6.4)–(6.6) becomes

$$\partial_t u - i(\partial_x^2 + \partial_y^2)u = \tilde f(u, \partial_x u, \partial_y u) \quad \text{if} \quad \delta = 1, \tag{6.7}$$

or

$$\partial_t u + 2i\partial_x\partial_y u = \tilde f(u, \partial_x u, \partial_y u) \quad \text{if} \quad \delta = -1,$$

where

$$\tilde f(u, \partial_x u, \partial_y u) = i\Big(\gamma - \frac{b}{2}\Big) f_0(u, \partial_x u, \partial_y u) + i\frac{b}{4} f_1(u, \partial_x u, \partial_y u) + i\frac{b}{4} f_2(u, \partial_x u, \partial_y u).$$

Thus we consider the nonlinear term $f(u, \partial_x u, \partial_y u)$ as in (6.1). Because $f(u, \partial_x u, \partial_y u)$ contains $\partial_x u$, $\partial_x \bar u$, $\partial_y u$ and $\partial_y \bar u$, the classical energy estimates are not available for the cases of the elliptic–hyperbolic and the hyperbolic–hyperbolic. Moreover, our method developed in Chapter 2 is not applicable. More precisely, the diagonalization technique does not work well, because of the non-locality of $f_1(u, \partial_x u, \partial_y u)$ and $f_2(u, \partial_x u, \partial_y u)$.

Up to recently, there is no mathematical result on the initial value problem for (6.7)–(6.2). First, F. Linares and G. Ponce ([34]) proved the local existence of small solutions to the initial value problems for the cases of the elliptic–hyperbolic and the hyperbolic–hyperbolic by the sharp smoothing estimates on $e^{it(\partial_x^2+\partial_y^2)}$ and $e^{-2it\partial_x\partial_y}$, which are basically due to C. E. Kenig, G. Ponce and L. Vega ([30]). Recently, using so-called abstract Cauchy–Kowalewski theorem, N. Hayashi and J.-C. Saut ([22]) have shown the local and the global existence of analytic solutions to the initial value problems for the cases of the elliptic–hyperbolic and the hyperbolic–hyperbolic.

The purpose of this chapter is to show the global existence of small amplitude solutions to (6.1)–(6.2). The main results are the following.

Theorem 6.1.1 (Local Existence) *Let m_6 be a sufficiently large integer. We put $a = \max(|a_1|, |a_2|)$. Then for any*

$$u_0 \in H^m \quad (m \in \mathbb{N} \ge m_6) \quad \textit{satisfying} \quad \|u_0\|_{L^2} < \frac{1}{2\sqrt{ae}}, \tag{6.8}$$

there exists a time $T = T(\|u_0\|_{m_1}) > 0$ such that the initial value problem (6.1)–(6.2) possesses a unique solution

$$u \in C_w\Big([0,T); H^m\Big) \cap C\Big([0,T); H^{m-1}\Big).$$

Theorem 6.1.2 (Global existence) *Let m_7 be a sufficiently large integer $\geq m_6 + 1$. Then there exists a constant $\delta > 0$ such that for any*

$$u_0 \in \bigcap_{j=0}^{5} H^{m-j,j} \quad (m \in \mathbb{N} \geq m_7 + 3) \quad \text{satisfying} \quad \sum_{j=0}^{5} \|u_0\|_{H^{m_2-j,j}} \leq \delta,$$

the initial value problem (6.1)–(6.2) *possesses a unique solution*

$$u \in \bigcap_{j=0}^{5}\Big(C_W\big([0,\infty); H^{m-j,j}\big) \cap C\big([0,\infty); H^{m-1-j,j}\big)\Big).$$

Remark 6.1.1 Since our analysis is based on the symbolic calculus of pseudo-differential operators, it is troublesome to determine the minimum of m_6 and m_7.

Remark 6.1.2 Our method is useless for the global existence of solutions to the case of the hyperbolic–hyperbolic.

Remark 6.1.3 In [13], using the inverse scattering technique, A. S. Fokas and L. Y. Sung proved the global existence for (6.3)–(6.4)–(6.6) with large initial data under the conditions $\delta = +1$, $c = -1$ and $2\gamma + b = 0$.

Now we explain the idea of the proofs. Theorem 6.1.1 follows from the energy inequality. Theorem 6.1.2 is proved by the *a priori* estimates which consist of the energy and the decay estimates.

For the energy estimates, we make use of S. Doi's method ([11]) for linear Schrödinger type equations

$$\partial_t u - i\Delta u + \sum_{j=1}^{N} b_j(t,x)\partial_j u + c(t,x)u = f(t,x) \quad (0,T)\times\mathbb{R}^N,$$

where $\partial_j = \partial/\partial x_j\ (j = 1,\cdots,N)$, $\nabla = (\partial = 1,\cdots,\partial_N)$, $\Delta = \nabla\cdot\nabla = \partial_1^2 + \cdots + \partial_N^2$ and $b_j(t,x), c(t,x) \in C\big([0,T]; \mathcal{B}^\infty(\mathbb{R}^N)\big)$. Roughly speaking, under the appropriate condition on $\operatorname{Im} b_j(t,x)$, there exists a automorphic $u \longmapsto Ku$ in $L^2(\mathbb{R}^N)$ such that $[K, -i\Delta]K^{-1}$ is elliptic which is stronger than $\sum_{j=1}^N \operatorname{Im} b_j(t,x)\partial_j$. Because one can choose $[K,-i\Delta]K^{-1}$ as sufficiently strong, $([K,-i\Delta]K^{-1} + \sum_{j=1}^N \operatorname{Im} b_j(t,x)\partial_j)^{1/2}$ gives the smoothing estimates of order $1/2$. We use this property to get the energy estimates for (6.1)–(6.2).

We explain the outline of the decay estimates. Because $f(u,\partial_x u,\partial_y u)$ satisfies the gauge invariance in a sense, J_x and J_y act well on $f(u,\partial_x u,\partial_y u)$. Then we can use J_x and J_y positively. Combining them and the Gagliardo–Nirenberg inequalities, we get the decay estimates. $f_1(u,\partial_x u,\partial_y u)$ and $f_2(u,\partial_x u,\partial_y u)$ behave as if they were cubic terms in one space dimension because of their nonlocality. This implies

$$f_1(u,\partial_x u,\partial_y u), f_2(u,\partial_x u,\partial_y u) = O\big((1+t)^{-1}\big)$$

as $t \to +\infty$. Then we need an extra time-decay. Fortunately we can use the null gauge condition of Y. Tsutsumi ([50])

$$\partial_x(|u|^2) = \frac{1}{2i(1+t)}\big(J_x u\bar{u} - u\overline{J_x u}\big) \tag{6.9}$$

and we can get the extra time-decay. (see also [27]).

The organizations of this chapter is as follows. Section 6.2 is devoted to obtain the smoothing effect of $e^{it(\partial_x^2+\partial_y^2)}$. Section 6.3 contains preliminary results. In Sections 6.4 and 6.5 we prove Theorems 6.1.1 and 6.1.2 respectively.

6.2 Linear Estimates for the Davey–Stewartson equation

In this section, following S. Doi [11], we obtain the smoothing effect of $e^{it(\partial_x^2+\partial_y^2)}$. We use the symbolic calculus of pseudo-differential operators on $\mathbb{R}$ and not on $\mathbb{R}^2$. We suppose $t \in [0,T]$ with some constant time $T > 0$. We define the pseudo-differential operators $K(t) = k(t,x,D_x)$ and $H(t) = h(t,y,D_y)$ by

$$\begin{aligned} k(t,x,\xi) &= \exp\Big(-\int_0^x \phi(t,s)ds\,\xi\langle\xi\rangle^{-1}\Big), \qquad k'(t,x,\xi) = k(t,x,\xi)^{-1},\\ h(t,y,\zeta) &= \exp\Big(-\int_0^y \psi(t,s)ds\,\zeta\langle\zeta\rangle^{-1}\Big), \qquad h'(t,y,\zeta) = h(t,y,\zeta)^{-1},\end{aligned}$$

$$\phi(t,s), \psi(t,s) \in C^1\big([0,T]; L^1(\mathbb{R})\big) \cap C\big([0,T]; \mathcal{B}^\infty(\mathbb{R})\big),$$

$$\phi(t,s), \psi(t,s) \geq 0 \quad \text{for} \quad (t,s) \in [0,T]\times\mathbb{R}.$$

For the sake of convenience, we put

$$\begin{aligned} B_K(t) &= \sup_{(x,\xi)\in\mathbb{R}^2} \sum_{\alpha+\beta\leq l}\Big(|\partial_x^\beta\partial_\xi^\alpha k(t,x,\xi)| + |\partial_x^\beta\partial_\xi^\alpha k'(t,x,\xi)|\Big),\\ B_H(t) &= \sup_{(y,\zeta)\in\mathbb{R}^2} \sum_{\alpha+\beta\leq l}\Big(|\partial_y^\beta\partial_\zeta^\alpha h(t,y,\zeta)| + |\partial_y^\beta\partial_\zeta^\alpha h'(t,y,\zeta)|\Big),\\ B_\phi^0(t) &= \int_{-\infty}^{+\infty}\phi(t,s)ds, \qquad B_\psi^0(t) = \int_{-\infty}^{+\infty}\psi(t,s)ds,\\ B_\phi^1(t) &= \sup_{x\in\mathbb{R}}\Big|\int_0^x \partial_t\phi(t,s)ds\Big|, \quad B_\psi^1(t) = \sup_{y\in\mathbb{R}}\Big|\int_0^y \partial_t\phi(t,s)ds\Big|,\\ B_\phi^\infty(t) &= \sup_{x\in\mathbb{R}}\sum_{\alpha\leq l}\Big(|\partial_x^\alpha\phi(t,x)| + |\partial_t\partial_x^\alpha\phi(t,x)|\Big),\\ B_\psi^\infty(t) &= \sup_{y\in\mathbb{R}}\sum_{\alpha\leq l}\Big(|\partial_y^\alpha\psi(t,y)| + |\partial_t\partial_y^\alpha\psi(t,y)|\Big)\end{aligned}$$

where $l \in \mathbb{N}$ is some large integer (see Remark 6.1.1). $K(t)$ and $H(t)$ are the automorphic on $L^2(\mathbb{R}^2)$ in some sense.

Lemma 6.2.1

$$\begin{aligned} \|u\| &\le CB_K(t)^2\Big(1+B^0_\phi(t)\Big)\Big(1+B^\infty_\phi(t)\Big) \\ &\quad\times\Big(\|K(t)u\|+\|u\|_{-1}\Big), &(6.10)\\ \Big(\|K(t)u\|+\|u\|_{-1}\Big) &\le CB_K(t)\|u\|, &(6.11)\\ \|u\| &\le CB_H(t)^2\Big(1+B^0_\psi(t)\Big)\Big(1+B^\infty_\psi(t)\Big) \\ &\quad\times\Big(\|H(t)u\|+\|u\|_{-1}\Big), &(6.12)\\ \Big(\|H(t)u\|+\|u\|_{-1}\Big), &\le CB_H(t)\|u\|, &(6.13)\end{aligned}$$

for $u \in L^2(\mathbb{R}^2)$ and $t \in [0,T]$.

Proof. See Lemma 2.3.3 ∎

Now we obtain the smoothing estimates.

Lemma 6.2.2 *We put $f^\varepsilon = \partial_t u - (i+\varepsilon)(\partial_x^2+\partial_y^2)u$, $(\varepsilon \in [0,1])$. Then there exists a constant $C_1 > 0$ which is independent of $\varepsilon \in [0,1]$, such that*

$$\begin{aligned} \frac{d}{dt}\|K(t)u(t)\|^2 &\le -4\Big(\phi(t,x)\langle D_x\rangle^{1/2}K(t)u(t), \langle D_x\rangle^{1/2}K(t)u(t)\Big) \\ &- 2\varepsilon\Big(\|\partial_x K(t)u(t)\|^2+\|\partial_y K(t)u(t)\|^2\Big) \\ &+ C_1\Big(B^1_\phi(t)+B^\infty_\phi(t)\Big)B_{etc}(t)\Big(\|K(t)u(t)\|+\|u(t)\|_{-1}\Big)^2 \\ &+ 2Re\Big(K(t)f^\varepsilon(t), K(t)u(t)\Big), &(6.14)\\ \frac{d}{dt}\|H(t)u(t)\|^2 &\le -4\Big(\psi(t,y)\langle D_y\rangle^{1/2}H(t)u(t), \langle D_y\rangle^{1/2}H(t)u(t)\Big) \\ &- 2\varepsilon\Big(\|\partial_x H(t)u(t)\|^2+\|\partial_y H(t)u(t)\|^2\Big) \\ &+ C_1\Big(B^1_\psi(t)+B^\infty_\psi(t)\Big)B'_{etc}(t)\Big(\|H(t)u(t)\|+\|u(t)\|_{-1}\Big)^2 \\ &+ 2Re\Big(H(t)f^\varepsilon(t), H(t)u(t)\Big), &(6.15)\end{aligned}$$

for $u \in C\Big([0,T];H^2\Big)\cap C^1\Big([0,T];L^2\Big)$ and $t \in [0,T]$, where

$$\begin{aligned} B_{etc}(t) &= B_K(t)^3\Big(1+B^\infty_\phi(t)\Big)^3\Big(1+B^0_\phi(t)\Big), \\ B'_{etc}(t) &= B_H(t)^3\Big(1+B^\infty_\psi(t)\Big)^3\Big(1+B^0_\psi(t)\Big). \end{aligned}$$

Proof. We show (6.14). The simple calculations yield

$$\begin{aligned} K(t)f^\varepsilon &= \partial_t\Big(K(t)u\Big) - (i+\varepsilon)(\partial_x^2+\partial_y^2)\Big(K(t)u\Big) \\ &+ 2(1-i\varepsilon)\langle D_x\rangle^{1/2}\phi(t,x)\langle D_x\rangle^{1/2}\Big(K(t)u\Big) + R^\varepsilon_1(t)u, &(6.16)\end{aligned}$$

$$\begin{aligned} R^\varepsilon_1(t) &= R_2(t) - (i+\varepsilon)R_3(t), \\ \sigma\Big(R_2(t)\Big)(x,\xi) &= -\partial_t k(t,x,\xi) = \int_0^x \partial_t\phi(t,s)ds\,\xi\langle\xi\rangle^{-1}k(t,x,\xi), \\ R_3(t) &= \Big[K(t),\partial_x^2\Big] - 2i\langle D_x\rangle^{1/2}\phi(t,x)\langle D_x\rangle^{1/2} \\ &= R_4(t)+R_5(t)+R_6(t)K(t), \\ \sigma\Big(R_4(t)\Big)(x,\xi) &= \Big(-2i\phi(t,x)\langle\xi\rangle^{-1} - \phi(t,x)^2\xi^2\langle\xi\rangle^{-2} + \partial_x\phi(t,x)\xi\langle\xi\rangle^{-1}\Big)k(t,x,\xi), \\ \sigma\Big(R_5(t)\Big)(x,\xi) &= -\frac{i}{2\pi}\xi\langle\xi\rangle^{-1}\int_0^1\iint_{\mathbb{R}\times\mathbb{R}} e^{-iz\eta}(\xi+\theta\eta)\langle\xi+\theta\eta\rangle^{-1} \\ &\quad\times\phi(t,x+z)k(t,x+z,\xi)dzd\eta d\theta, \\ \sigma\Big(R_6(t)\Big)(x,\xi) &= -\frac{1}{2\pi}\langle\xi\rangle^{1/2}\int_0^1\iint_{\mathbb{R}\times\mathbb{R}} e^{-iz\eta}(\xi+\theta\eta)\langle\xi+\theta\eta\rangle^{-3/2} \\ &\quad\times\partial_x\phi(t,x+z)dzd\eta d\theta. \end{aligned}$$

It is easy to see that there exists a constant $C'_1 > 0$ which is independent of $\varepsilon \in [0,1]$, such that

$$\Big\|R^\varepsilon_1(t)\Big\|_{\mathcal{L}(L^2)} \le C'_1\Big(B^1_\phi(t)+B^\infty_\phi(t)\Big)\Big(1+B^\infty_\phi(t)\Big)B_K(t) \tag{6.17}$$

for $t \in [0,T]$ and $\varepsilon \in [0,1]$. (6.10), (6.16) and (6.17) imply (6.14). In the same way we can get (6.15). ∎

6.3 Preliminaries for the Davey–Stewartson equation

This section is devoted to the estimates of the nonlinear term $f(u,\partial_x u,\partial_y u)$. In particular, the Gagliardo–Nirenberg inequalities

$$\begin{aligned} \|v\|_{L^\infty(\mathbb{R})} &\le C\|\partial_x v\|^{1/2}_{L^2(\mathbb{R})}\|v\|^{1/2}_{L^2(\mathbb{R})} \quad \text{for } v \in H^1(\mathbb{R}), &(6.18)\\ \|u\|_{L^\infty(\mathbb{R})} &\le C\sum_{\alpha+\beta=2}\|\partial_x^\alpha\partial_y^\beta u\|^{1/2}_{L^2(\mathbb{R}^2)}\|u\|^{1/2}_{L^2(\mathbb{R}^2)} \quad \text{for } u \in H^2(\mathbb{R}^2), &(6.19)\end{aligned}$$

play important roles to get the decay estimates of solutions.

Let l be the same integer as in the previous section. We put

$$\phi(x) = M\int_{-\infty}^{+\infty}\Big|u(x,y)\Big|^2dy, \quad \psi(y) = M\int_{-\infty}^{+\infty}\Big|u(x,y)\Big|^2dx,$$

with some constant $M > 0$. Similarly we define $K = k(x, D_x)$, $K' = k'(x, D_x)$, $H = h(x, D_x)$ and $H' = h'(x, D_x)$ by

$$k(x,\xi) = \exp\left(-\int_0^x \phi(x')dx'\xi\langle\xi\rangle^{-1}\right), \qquad k'(x,\xi) = k(x,\xi)^{-1},$$
$$h(y,\zeta) = \exp\left(-\int_0^y \psi(y')dy'\zeta\langle\zeta\rangle^{-1}\right), \qquad h'(y,\zeta) = h(y,\zeta)^{-1}.$$

We put $R_0 = K'K - 1$ and we introduce B_K, B_H, B_ϕ^∞ and B_ψ^∞. To resolve the loss of derivatives, we prepare

Lemma 6.3.1 *Let m be an integer $\geq l+1$. Then we have*

$$\begin{aligned}
&\left|\left(K\left\{\left(\int_y^{+\infty} u'\partial_x P v dy'\right)u\right\}, Kw\right)\right| \\
&\leq \frac{1}{2M}\Big(1 + \sup_{(x,\xi)\in\mathbb{R}^2}\big|k(x,\xi)p(x,\xi)\big|^2\Big)\big(\phi(x)\langle D_x\rangle^{1/2}Kw, \langle D_x\rangle^{1/2}Kw\big) \\
&+ CB_K^4\int_{-\infty}^{+\infty}\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y)\big|^2 dy\|Kw\|^2
\end{aligned} \tag{6.20}$$

for $u \in H^m$ and $w \in H^1$, where

$$u' = \bar{u}, P = K', v = Kw \quad \text{or} \quad u' = u, P = K, v = \overline{Kw},$$

and

$$\begin{aligned}
&\left|\left(H\left\{\left(\int_x^{+\infty} u'\partial_y Q v dx'\right)u\right\}, Hw\right)\right| \\
&\leq \frac{1}{2M}\Big(1 + \sup_{(y,\zeta)\in\mathbb{R}^2}\big|h(y,\zeta)q(y,\zeta)\big|^2\Big)\big(\psi(x)\langle D_y\rangle^{1/2}Hw, \langle D_y\rangle^{1/2}Hw\big) \\
&+ CB_H^4\int_{-\infty}^{+\infty}\sup_{y\in\mathbb{R}}\big|\langle D_y\rangle^l u(x,y)\big|^2 dx\|Hw\|^2
\end{aligned} \tag{6.21}$$

for $u \in H^m$ and $w \in H^1$, where

$$u' = \bar{u}, P = H', v = Hw \quad \text{or} \quad u' = u, P = H, v = \overline{Hw},$$

Proof. We have only to show (6.20). The simple calculation yields

$$\begin{aligned}
&\left|\left(K\left\{\left(\int_y^{+\infty} u'\partial_x P v dy'\right)u\right\}, Kw\right)\right| \\
&\leq \left|\left(\int_y^{+\infty} r_8(y,y',x,D_x)v dy', Kw\right)\right| \\
&+ \left|\left(u' r_9(x,D_x)\langle D_x\rangle^{1/2} v dy', \bar{u}\langle D_x\rangle^{1/2}Kw\right)\right|,
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
r_8(y,y',x,D_x) &= \big[K, u(x,y)u'(x,y)\big]\partial_x P \\
&+ \big[u(x,y)u'(x,y')K\partial_x P, \langle D_x\rangle^{1/2}\big]\langle D_x\rangle^{-1/2} \\
&+ \langle D_x\rangle^{1/2}u(x,y)u'(x,y')\big(K\partial_x P\langle D_x\rangle^{-1/2} - r_9(x,D_x)\langle D_x\rangle^{1/2}\big), \\
r_9(x,\xi) &= ik(x,\xi)p(x,\xi)\xi\langle\xi\rangle^{-1}.
\end{aligned}$$

It is easy to see

$$\|r_8(y,y',\cdot,\cdot)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq CB_K^2\Big(\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y)\big|\Big)\Big(\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y')\big|\Big).$$

Then we have

$$\begin{aligned}
&\left|\left(\int_y^{+\infty} r_8(y,y',x,D_x)v dy', Kw\right)\right| \\
&\leq \iiint\big|r_8(y,y',x,D_x)v(x,y')\big|\big|Kw(x,y)\big|dxdydy' \\
&\leq \iint\left(\int\big|r_8(y,y',x,D_x)v(x,y')\big|^2dx\right)^{1/2}\left(\int\big|Kw(x,y)\big|^2dx\right)^{1/2} \\
&\qquad\times\Big(\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y)\big|\Big)\Big(\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y')\big|\Big)dydy' \\
&\leq CB_K^2\left\{\int\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y)\big|\left(\int\big|Kw(x,y)\big|^2dx\right)^{1/2}dy\right\}^2 \\
&\leq CB_K^2\left(\int\sup_{x\in\mathbb{R}}\big|\langle D_x\rangle^l u(x,y)\big|^2dy\right)\|Kw\|^2.
\end{aligned} \tag{6.23}$$

On the other hand, we get

$$\begin{aligned}
&\left|\left(\int u' r_9(x,D_x)\langle D_x\rangle^{1/2}v dy', \bar{u}\langle D_x\rangle^{1/2}Kw\right)\right| \\
&\leq \iiint\big|u(x,y')\big|\big|r_9(x,D_x)\langle D_x\rangle^{1/2}v(x,y')\big|\big|u(x,y)\big|\big|\langle D_x\rangle^{1/2}Kw(x,y)\big|dxdydy' \\
&\leq \int\left(\int\big|u(x,y')\big|^2dy'\right)^{1/2}\left(\int\big|r_9(x,D_x)\langle D_x\rangle^{1/2}v(x,y')\big|^2dy'\right)^{1/2} \\
&\qquad\times\left(\int\big|u(x,y)\big|^2dy\right)^{1/2}\left(\int\big|\langle D_x\rangle^{1/2}Kw(x,y)\big|^2dy\right)^{1/2}dx \\
&= \frac{1}{M}\int\phi(x)\left(\int\big|r_9(x,D_x)\langle D_x\rangle^{1/2}v(x,y')\big|^2dy'\right)^{1/2}\left(\int\big|\langle D_x\rangle^{1/2}Kw(x,y)\big|^2dy\right)^{1/2}dx \\
&\leq \frac{1}{M}\big\|\phi(x)^{1/2}r_9(x,D_x)\langle D_x\rangle^{1/2}v\big\|\big\|\phi(x)^{1/2}\langle D_x\rangle^{1/2}Kw\big\| \\
&\leq \frac{1}{2M}\Big\{\big\|\phi(x)^{1/2}r_9(x,D_x)\langle D_x\rangle^{1/2}v\big\|^2 + \big\|\phi(x)^{1/2}\langle D_x\rangle^{1/2}Kw\big\|^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2M}\Big\{\mathrm{Re}\,\Big(r_9(x,D_x)^*\phi(x)r_9(x,D_x)\langle D_x\rangle^{1/2}v,\ \langle dD_x 1/2v\Big) \\
&\qquad\qquad +\Big(\phi(x)\langle D_x\rangle^{1/2}Kw,\ \langle D_x\rangle^{1/2}Kw\Big)\Big\} \\
&\le \frac{1}{2M}\Big\{\mathrm{Re}\,\Big(\phi(x)r_{10}(x,D_x)\langle D_x\rangle^{1/2}v,\ \langle D_x\rangle^{1/2}v\Big) \\
&\qquad\qquad +\Big(\phi(x)\langle D_x\rangle^{1/2}Kw,\ \langle D_x\rangle^{1/2}Kw\Big)\Big\} + CB_\phi^\infty B_K^4\|Kw\|^2,
\end{aligned}
$$

where

$$
r_{10}(x,\xi) = \Big|k(x,\xi)p(x,\xi)\Big|^2\xi^2\langle\xi\rangle^{-2}.
$$

Then the sharp Gårding inequality implies

$$
\begin{aligned}
&\Big|\Big(\int u' r_9(x,D_x)\langle D_x\rangle^{1/2}v\,dy',\ \bar u\langle D_x\rangle^{1/2}Kw\Big)\Big| \\
\le\ & \frac{1}{2M}\Big\{1+\sup_{(x,\xi)\in\mathbb{R}^2}\Big|k(x,\xi)p(x,\xi)\Big|^2\Big\}\Big(\phi(x)\langle D_x\rangle^{1/2}Kw,\ \langle D_x\rangle^{1/2}Kw\Big) \\
+\ & CB_\phi^\infty B_K^4\|Kw\|^2 \\
\le\ & \frac{1}{2M}\Big\{1+\sup_{(x,\xi)\in\mathbb{R}^2}\Big|k(x,\xi)p(x,\xi)\Big|^2\Big\}\Big(\phi(x)\langle D_x\rangle^{1/2}Kw,\ \langle D_x\rangle^{1/2}Kw\Big) \\
+\ & C\left(\int\sup_{x\in\mathbb{R}}\Big|\langle D_x\rangle^l u(x,y)\Big|^2 dy\right)B_K^4\|Kw\|^2.
\end{aligned}
\tag{6.24}
$$

Substituting (6.23) and (6.24) into (6.22), we obtain (6.20). ∎

To prove Theorem 6.1.1, we prepare the following two lemmata.

Lemma 6.3.2 *Let m be an integer ≥ 2. Then there exists a constant $C>0$ depending only on m such that*

$$
\begin{aligned}
\Big\|f_0(u,\partial_x u,\partial_y u)\Big\|_m &+ \sum_{\substack{\alpha+\beta\le m\\ \alpha\le m-1}}\Big\|\partial_x^\alpha\partial_y^\beta\Big(f_1(u,\partial_x u,\partial_y u)\Big)\Big\| \\
&+ \sum_{\substack{\alpha+\beta\le m\\ \beta\le m-1}}\Big\|\partial_x^\alpha\partial_y^\beta\Big(f_2(u,\partial_x u,\partial_y u)\Big)\Big\| \le C\|u\|^2_{[\frac{m+1}{2}]+1}\|u\|_m,
\end{aligned}
\tag{6.25}
$$

$$
\begin{aligned}
&\sum_{j=0}^2\Big\|f_j(u,\partial_x u,\partial_y u)-f_j(v,\partial_x v,\partial_y v)\Big\|_{m-1} \\
\le\ & C\Big(\|u\|_m+\|v\|_m\Big)^2\|u-v\|_m,
\end{aligned}
\tag{6.26}
$$

for $u,v\in H^m$, and

$$
\begin{aligned}
&\left\|\partial_x^m\Big(f_1(u,\partial_x u,\partial_y u)\Big)-\Big(\int_y^{+\infty}\bar u\partial_x^{m+1}u\,dy'\Big)u-\Big(\int_y^{+\infty}u\partial_x^{m+1}\bar u\,dy'\Big)u\right\| \\
+\ &\left\|\partial_y^m\Big(f_2(u,\partial_x u,\partial_y u)\Big)-\Big(\int_x^{+\infty}\bar u\partial_y^{m+1}u\,dx'\Big)u-\Big(\int_x^{+\infty}u\partial_y^{m+1}\bar u\,dx'\Big)u\right\| \\
\le\ & C\|u\|^2_{[\frac{m+1}{2}]+1}\|u\|_m
\end{aligned}
\tag{6.27}
$$

for $u\in H^{m+1}$.

Lemma 6.3.3 *Let m be an integer $\geq l+1$. Then*

$$
\begin{aligned}
&\left|\left(K\left[\Big\{\int_y^{+\infty}(\bar u\partial_x^{m+1}u+u\partial_x^{m+1}\bar u)dy'\Big\}u\right],K\partial_x^m u\right)\right| \\
\le\ & \frac{1}{2M}\Big(3+e^{4M\|u\|^2}\Big)\Big(\phi(x)\langle D_x\rangle^{1/2}K\partial_x^m u,\ \langle D_x\rangle^{1/2}K\partial_x^m u\Big) \\
+\ & CB_K^5\|u\|^2_{l+1}\|K\partial_x^m u\|\Big(\|K\partial_x^m u\|+\|u\|_{m-1}\Big),
\end{aligned}
\tag{6.28}
$$

$$
\begin{aligned}
&\left|\left(H\left[\Big\{\int_y^{+\infty}(\bar u\partial_y^{m+1}u+u\partial_y^{m+1}\bar u)dy'\Big\}u\right],H\partial_y^m u\right)\right| \\
\le\ & \frac{1}{2M}\Big(3+e^{4M\|u\|^2}\Big)\Big(\psi(x)\langle D_y\rangle^{1/2}H\partial_y^m u,\ \langle D_x\rangle^{1/2}H\partial_y^m u\Big) \\
+\ & CB_H^5\|u\|^2_{l+1}\|H\partial_y^m u\|\Big(\|H\partial_y^m u\|+\|u\|_{m-1}\Big),
\end{aligned}
\tag{6.29}
$$

for any $u\in H^{m+1}$.

Proof of Lemma 6.3.2. The Gagliardo–Nirenberg inequality implies

$$
\Big\|f_0(u,\partial_x u,\partial_y u)\Big\|_m \le C\|u\|^2_{L^\infty}\|u\|_m. \tag{6.30}
$$

Let α and β be non-negative integers satisfying $\alpha+\beta\le m$ and $\alpha\le m-1$. We have

$$
\begin{aligned}
\partial_x^\alpha\partial_y^\beta\Big(f_1(u,\partial_x u,\partial_y u)\Big) &= g_1^{\alpha\beta}(u)+g_2^{\alpha\beta}(u) \\
g_1^{\alpha\beta}(u) &= -\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha+1\\ \alpha_3\le\alpha}}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1} \\
&\quad \frac{\alpha!(\alpha_1+\alpha_2)}{\alpha_1!\alpha_2!\alpha_3!}\,\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)} \\
&\times\ \partial_x^{\alpha_1}\partial_y^{\beta_1}u\,\partial_x^{\alpha_2}\partial_y^{\beta_2}\bar u\,\partial_x^{\alpha_3}\partial_y^{\beta_3}u, \\
g_2^{\alpha\beta}(u) &= \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha+1\\ \alpha_3\le\alpha}}\frac{\alpha!(\alpha_1+\alpha_2)}{\alpha_1!\alpha_2!\alpha_3!} \\
&\times\ \Big(\int_y^{+\infty}\partial_x^{\alpha_1}u\,\partial_x^{\alpha_2}\bar u\,dy'\Big)\partial_x^{\alpha_3}\partial_y^\beta u.
\end{aligned}
$$

In the same way as (6.30), we get

$$\left\|g_1^{\alpha\beta}(u)\right\| \le C\|u\|_{L^\infty}^2\|u\|_m. \tag{6.31}$$

Using the Schwarz inequality with respect to y', we have

$$\begin{aligned}\left\|g_2^{\alpha\beta}(u)\right\| &\le \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha+1\\ \alpha_3\le\alpha}} \frac{\alpha!(\alpha_1+\alpha_2)}{\alpha_1!\alpha_2!\alpha_3!}\\ &\times \left\{\int\left(\int|\partial_x^{\alpha_1}u|^2dy_1\right)\left(\int|\partial_x^{\alpha_2}u|^2dy_2\right)\left(\int|\partial_x^{\alpha_3}\partial_y^{\beta}u|^2dy_3\right)dx\right\}^{1/2}\\ &\le C\left(\sup_{x\in\mathbb{R}}\int\left|\langle D_x;D_y\rangle^{[\frac{m+1}{2}]}\right|^2dy\right)\|u\|_m\\ &\le C\|u\|^2_{[\frac{m+1}{2}]+1}\|u\|_m. \end{aligned}\tag{6.32}$$

Combining (6.31) and (6.32), we obtain

$$\sum_{\substack{\alpha+\beta\le m\\ \alpha\le m-1}}\left\|\partial_x^\alpha\partial_y^\beta\big(f_1(u,\partial_x u,\partial_y u)\big)\right\| \le C\|u\|^2_{[\frac{m+1}{2}]+1}\|u\|_m. \tag{6.33}$$

In the same way, we have

$$\sum_{\substack{\alpha+\beta\le m\\ \beta\le m-1}}\left\|\partial_x^\alpha\partial_y^\beta\big(f_2(u,\partial_x u,\partial_y u)\big)\right\| \le C\|u\|^2_{[\frac{m+1}{2}]+1}\|u\|_m. \tag{6.34}$$

(6.30), (6.33) and (6.34) show (6.25). Similarly we can obtain (6.26) and (6.27). ∎

Proof of Lemma 6.3.3. We here note

$$\partial_x^{m+1}u = \partial_x K'K\partial_x^m u - \partial_x R_0\partial_x^m u, \quad \partial_x^{m+1}\bar u = \partial_x K\overline{K\partial_x^m u} - \overline{\partial_x R_0\partial_x^m u},$$

where $R_0 = K'K - 1 \in \mathcal{L}(H^{-3}, L^2)$. Using (6.20), we obtain (6.28). Similarly we get (6.29) by (6.21). ∎

Now we obtain the estimates on the nonlinear term $f(u,\partial_x u,\partial_y u)$ in order to prove Theorem 6.1.2. Using (6.19), we have

$$\|u\|_{L^\infty} \le C(1+t)^{-1}\sum_{\alpha'+\beta'=2}\|J_x^{\alpha'}J_y^{\beta'}u\|^{1/2}\|u\|^{1/2} \tag{6.35}$$

for $u \in H^2\cap H^{0,2}$. Let us introduce

$$X_{m,n}(t) = \sum_{\substack{\alpha+\beta\le m\\ \alpha'+\beta'=n}}\left\|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}u(t)\right\|, \quad m,n\in\mathbb{Z}_+.$$

We here remark the properties of J_x and J_y, those are

$$\left[\partial_t - i(\partial_x^2+\partial_y^2), J_x\right] = \left[\partial_t - i(\partial_x^2+\partial_y^2), J_y\right] = 0,$$

$$\left[J_x, J_y\right] = \left[\partial_x, J_y\right] = \left[\partial_y, J_x\right] = 0, \quad \left[\partial_x, J_x\right] = \left[\partial_y, J_y\right] = 1.$$

We show the following two lemmata.

Lemma 6.3.4 *Let m be an integer≥ 6. Then we have*

$$\begin{aligned} &\sum_{\substack{\alpha+\beta+\alpha'+\beta'\le m\\ \alpha'+\beta'\le 5}}\left\|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}\left\{f_0(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\|\\ &+ \sum_{\substack{\alpha+\beta+\alpha'+\beta'\le m\\ \alpha'+\beta'\le 5\\ \alpha+\alpha'\le m-1\\ \alpha'\le 4}}\left\|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}\left\{f_1(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\|\\ &+ \sum_{\substack{\alpha+\beta+\alpha'+\beta'\le m\\ \alpha'+\beta'\le 5\\ \beta+\beta'\le m-1\\ \beta'\le 4}}\left\|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'}J_y^{\beta'}\left\{f_2(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\|\\ &\le C(1+t)^{-2}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\sum_{j=0}^{5}X_{m-j,j}(t), \end{aligned}\tag{6.36}$$

$$\begin{aligned} &\sum_{\substack{\alpha+\alpha'=m\\ \alpha'\le 5}}\left\|\partial_x^\alpha J_x^{\alpha'}\left\{f_1(u(t),\partial_x u(t),\partial_y u(t))\right\} - \tilde f_1^{\alpha\alpha'}(u(t))u(t)\right\|\\ &+ \sum_{\substack{\beta+\beta'=m\\ \beta'\le 5}}\left\|\partial_x^\beta J_x^{\beta'}\left\{f_2(u(t),\partial_x u(t),\partial_y u(t))\right\} - \tilde f_2^{\beta\beta'}(u(t))u(t)\right\|\\ &+ \sum_{\alpha+\beta\le m-6}\left\|\partial_x^\alpha\partial_y^\beta J_x^5\left\{f_1(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\|\\ &+ \sum_{\alpha+\beta\le m-6}\left\|\partial_x^\alpha\partial_y^\beta J_y^5\left\{f_2(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\|\\ &\le C(1+t)^{-1}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\sum_{j=0}^{5}X_{m-j,j}(t), \end{aligned}\tag{6.37}$$

where

$$\begin{aligned}\tilde f_1^{\alpha\alpha'}(u) &= \int_y^{+\infty}\left(\partial_x^{\alpha+1}J_x^{\alpha'}u\bar u + (-1)^{\alpha'}u\overline{\partial_x^{\alpha+1}J_x^{\alpha'}u}\right)dy',\\ \tilde f_2^{\beta\beta'}(u) &= \int_x^{+\infty}\left(\partial_y^{\beta+1}J_y^{\beta'}u\bar u + (-1)^{\beta'}u\overline{\partial_y^{\beta+1}J_y^{\beta'}u}\right)dx'.\end{aligned}$$

Lemma 6.3.5 *Let m be an integer$\geq \max(l+1,6)$. Then we have*

$$\begin{aligned}
&\sum_{\substack{\alpha+\alpha'=m\\ \alpha'\leq 5}}\left|\Big(K(t)\big(\tilde{f_1}^{\alpha\alpha'}(u(t))u(t)\big),K(t)\partial_x^{\alpha}J_x^{\alpha'}u(t)\Big)\right|\\
\leq\ & \frac{1}{2M}\Big(3+e^{4M\|u(t)\|^2}\Big)\\
\times\ & \Big(\phi(t,x)\langle D_x\rangle^{1/2}K(t)\big(\partial_x^{\alpha}J_x^{\alpha'}u(t)\big),\langle D_x\rangle^{1/2}K(t)\big(\partial_x^{\alpha}J_x^{\alpha'}u(t)\big),\Big)\\
+\ & CB_K(t)^4(1+t)^{-1}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\left\|K(t)\big(\partial_x^{\alpha}J_x^{\alpha'}u(t)\big)\right\|^2\\
+\ & CB_K(t)^3B_\phi^\infty(t)(1+t)^{-1}\\
\times\ & \left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\left\|K(t)\big(\partial_x^{\alpha}J_x^{\alpha'}u(t)\big)\right\|\left\|\partial_x^{\alpha}J_x^{\alpha'}u(t)\right\|, \qquad (6.38)\\
&\sum_{\substack{\beta+\beta'=m\\ \beta'\leq 5}}\left|\Big(H(t)\big(\tilde{f_2}^{\beta\beta'}(u(t))u(t)\big),H(t)\partial_y^{\beta}J_y^{\beta'}u(t)\Big)\right|\\
\leq\ & \frac{1}{2M}\Big(3+e^{4M\|u(t)\|^2}\Big)\\
\times\ & \Big(\psi(t,y)\langle D_y\rangle^{1/2}H(t)\big(\partial_y^{\beta}J_y^{\beta'}u(t)\big),\langle D_y\rangle^{1/2}H(t)\big(\partial_y^{\beta}J_y^{\beta'}u(t)\big),\Big) \qquad (6.39)\\
+\ & CB_H(t)^4(1+t)^{-1}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\left\|H(t)\big(\partial_y^{\beta}J_y^{\beta'}u(t)\big)\right\|^2\\
+\ & CB_H(t)^3B_\psi^\infty(t)(1+t)^{-1}\\
\times\ & \left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\left\|H(t)\big(\partial_y^{\beta}J_y^{\beta'}u(t)\big)\right\|\left\|\partial_y^{\beta}J_y^{\beta'}u(t)\right\|, \qquad (6.40)
\end{aligned}$$

$$\textit{for}\quad t\in[0,T],\quad u\in\bigcap_{j=0}^{5}C\Big([0,T];H^{m+1-j,j}\Big).$$

Proof of Lemma 6.3.4. It is easy to see

$$\begin{aligned}
&\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leq m\\ \alpha'+\beta'\leq 5}}\left\|\partial_x^{\alpha}\partial_y^{\beta}J_x^{\alpha'}J_y^{\beta'}\big\{f_0(u(t),\partial_xu(t),\partial_yu(t))\big\}\right\|\\
\leq\ & C(1+t)^{-2}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\sum_{j=0}^{5}X_{m-j,j}(t). \qquad (6.41)
\end{aligned}$$

Let α, β, α' and β' be non-negative integers satisfying $\alpha+\beta+\alpha'+\beta'\leq m$, $\alpha'+\beta'\leq 5$, $\alpha+\alpha'\leq m-1$ and $\alpha'\leq 4$. We have

$$\partial_x^{\alpha}\partial_y^{\beta}J_x^{\alpha'}J_y^{\beta'}\big(f_1(u,\partial_xu,\partial_yu)\big)=g_3^{\alpha\beta\alpha'\beta'}(u)+g_4^{\alpha\beta\alpha'\beta'}(u),$$

$$\begin{aligned}
g_3^{\alpha\beta\alpha'\beta'}(u)\ =\ & -\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\alpha_1'+\alpha_2'+\alpha_3'=\alpha'+1}\ \sum_{\beta_1+\beta_2+\beta_3=\beta}\ \sum_{\beta_1'+\beta_2'+\beta_3'=\beta'-1}\\
\times\ & \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha_1'+\alpha_2')}{\alpha_1'!\alpha_2'!\alpha_3'!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!}\frac{\beta'!}{\beta_1'!\beta_2'!\beta_3'!(\beta'-\beta_3')}(-1)^{\alpha_2'+\beta_2'}\\
\times\ & \partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha_1'}J_y^{\beta_1'}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha_2'}J_y^{\beta_2'}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha_3'}J_y^{\beta_3'}u\\
-\ & \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\alpha_1'+\alpha_2'+\alpha_3'=\alpha'}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1}\\
\times\ & \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha_1'+\alpha_2')}{\alpha_1'!\alpha_2'!\alpha_3'!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}(-1)^{\alpha_2'}\\
\times\ & \partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha_1'}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha_2'}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha_3'}J_y^{\beta'}u\\
+\ & \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\substack{\alpha_1'+\alpha_2'+\alpha_3'=\alpha'\\ \alpha_1'\geq 1}}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1}\\
\times\ & \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha_1'+\alpha_2')}{\alpha_1'!\alpha_2'!\alpha_3'!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}(-1)^{\alpha_2'}\\
\times\ & \alpha_1'\partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha_1'-1}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha_2'}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha_3'}J_y^{\beta'}u\\
+\ & \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\substack{\alpha_1'+\alpha_2'+\alpha_3'=\alpha'\\ \alpha_2'\geq 1}}\ \sum_{\beta_1+\beta_2+\beta_3=\beta-1}\\
\times\ & \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha_1'+\alpha_2')}{\alpha_1'!\alpha_2'!\alpha_3'!}\frac{\beta!}{\beta_1!\beta_2!\beta_3!(\beta-\beta_3)}(-1)^{\alpha_2'}\\
\times\ & \alpha_2'\partial_x^{\alpha_1}\partial_y^{\beta_1}J_x^{\alpha_1'-1}u\overline{\partial_x^{\alpha_2}\partial_y^{\beta_2}J_x^{\alpha_2'-1}u}\partial_x^{\alpha_3}\partial_y^{\beta_3}J_x^{\alpha_3'}J_y^{\beta'}u,\\
g_4^{\alpha\beta\alpha'\beta'}(u)\ =\ & \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\ \sum_{\alpha_1'+\alpha_2'+\alpha_3'=\alpha'+1}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}\frac{\alpha'!(\alpha_1'+\alpha_2')}{\alpha_1'!\alpha_2'!\alpha_3'!}\\
\times\ & (-1)^{\alpha_2'}\frac{1}{2i(1+t)}\\
\times\ & \left(\int_y^{+\infty}\partial_x^{\alpha_1}J_x^{\alpha_1'}u\overline{\partial_x^{\alpha_2}J_x^{\alpha_2'}u}dy'\right)\partial_x^{\alpha_3}\partial_y^{\beta}J_x^{\alpha_3'}J_y^{\beta'}u.
\end{aligned}$$

Here we used the null gauge condition (6.9) in the right hand side of $g_4^{\alpha\beta\alpha'\beta'}(u)$. In the same way as (6.41), we get

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\leq m\\ \alpha'+\beta'\leq 5\\ \alpha+\alpha'\leq m-1\\ \alpha'\leq 4}}\left\|g_3^{\alpha\beta\alpha'\beta'}(u(t))\right\|\leq C(1+t)^{-2}\left(\sum_{j=0}^{4}X_{m-j-1,j}(t)\right)^2\sum_{j=0}^{5}X_{m-j,j}(t). \qquad (6.42)$$

We here remark that (6.18) implies

$$\int \sup_{x\in\mathbb{R}} \left|u(t,x,y)\right|^2 dy \le C(1+t)^{-1} \sum_{\alpha+\beta\le 1} \left\|\partial_x^\alpha J_x^\beta u(t)\right\|^2. \tag{6.43}$$

(6.43) and the same calculations in (6.32) give

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta'\le m\\ \alpha'+\beta'\le 5\\ \alpha+\alpha'\le m-1\\ \alpha'\le 4}} \left\|g_4^{\alpha\beta\alpha'\beta'}(u(t))\right\| \le C(1+t)^{-2}\left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \sum_{j=0}^{5} X_{m-j,j}(t) \tag{6.44}$$

(6.42) and (6.44) show

$$\begin{aligned}
&\sum_{\substack{\alpha+\beta+\alpha'+\beta'\le m\\ \alpha'+\beta'\le 5\\ \alpha+\alpha'\le m-1\\ \alpha'\le 4}} \left\|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'} J_y^{\beta'}\left\{f_1(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\| \\
&\le C(1+t)^{-2}\left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \sum_{j=0}^{5} X_{m-j,j}(t).
\end{aligned} \tag{6.45}$$

Similarly we can get

$$\begin{aligned}
&\sum_{\substack{\alpha+\beta+\alpha'+\beta'\le m\\ \alpha'+\beta'\le 5\\ \beta+\beta'\le m-1\\ \beta'\le 4}} \left\|\partial_x^\alpha\partial_y^\beta J_x^{\alpha'} J_y^{\beta'}\left\{f_2(u(t),\partial_x u(t),\partial_y u(t))\right\}\right\| \\
&\le C(1+t)^{-2}\left(\sum_{j=0}^{4} X_{m-j-1,j}(t)\right)^2 \sum_{j=0}^{5} X_{m-j,j}(t).
\end{aligned} \tag{6.46}$$

Combining (6.41), (6.45) and (6.46), we obtain (6.36). The proof of (6.37) is almost same as that of (6.36). We here remark that we do not use the null gauge condition (6.9) to get (6.37). ∎

Proof of Lemma 6.3.5. The proof of is basically same as that of Lemma 6.3.3. Lemma 6.3.1 and (6.43) imply (6.38) and (6.40). ∎

6.4 Proof of local existence theorem

We prove Theorem 6.1.1 by the parabolic regularization and the uniform estimates which follow from Lemma 6.2.2. First we consider

$$\partial_t u^\varepsilon - (i+\varepsilon)(\partial_x^2+\partial_y^2)u^\varepsilon = f(u^\varepsilon,\partial_x u^\varepsilon,\partial_y u^\varepsilon) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \tag{6.47}$$

$$u^\varepsilon(0,x,y) = u_0(x,y) \quad \text{in} \quad \mathbb{R}^2, \tag{6.48}$$

where $\varepsilon\in(0,1]$. We remark that the initial data u_0 is independent of $\varepsilon\in(0,1]$. Since the elliptic term $-\varepsilon(\partial_x^2+\partial_y^2)$ resolve the loss of derivatives, we obtain

Lemma 6.4.1 *Let m be an integer ≥ 2. For any $u_0\in H^m$, there exists a time $T_\varepsilon = T\left(\varepsilon,\|u_0\|_2\right) > 0$ such that the initial value problem (6.47)–(6.48) possesses a unique solution $u\in C\left([0,T_\varepsilon);H^m\right)$. Moreover the mapping $u_0\longmapsto u$ is continuous between the above spaces.*

Proof. It is easy to show Lemma 6.4.1 by using (6.25) and (6.26). We omit the rigorous proof. ∎

Secondly we prove the existence of a solution to (6.1)–(6.2) by the uniform estimates on $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$. More precisely, we show that there exists a time $T>0$ which is independent $\varepsilon\in(0,1]$ such that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$. Let $m_6 = l+3$ where l is the same integer as in Section 6.2, and let m be an integer $\ge m_6$. We put $M = ae$. We define

$$\begin{aligned}
K^\varepsilon(t) &= k^\varepsilon(t,x,D_x),\\
k^\varepsilon(t,x,\xi) &= \exp\left(-\int_0^x \phi^\varepsilon(t,x')dx'\xi\langle\xi\rangle^{-1}\right),\\
\phi^\varepsilon(t,x) &= M\int\left|u^\varepsilon(t,x,y)\right|^2 dy,\\
H^\varepsilon(t) &= h^\varepsilon(t,y,D_y),\\
h^\varepsilon(t,x,\xi) &= \exp\left(-\int_0^y \psi^\varepsilon(t,y')dy'\zeta\langle\zeta\rangle^{-1}\right),\\
\psi^\varepsilon(t,y) &= M\int\left|u^\varepsilon(t,x,y)\right|^2 dx,
\end{aligned}$$

$$N_m^\varepsilon(t) = \sum_{\substack{\alpha+\beta\le m\\ \alpha,\beta\le m-1}} \left\|\partial_x^\alpha\partial_y^\beta u^\varepsilon(t)\right\| + \left\|K^\varepsilon(t)\partial_x^m u^\varepsilon(t)\right\| + \left\|H^\varepsilon(t)\partial_y^m u^\varepsilon(t)\right\|.$$

Since the initial data u_0 is independent of $\varepsilon\in(0,1]$, $N_m^\varepsilon(0)$ are also independent of $\varepsilon\in(0,1]$ and then we denote them by the same notation N_m. It is easy to see that there exists an increasing function $A(\cdot)$ on $[0,+\infty)$ such that

$$\begin{aligned}
&B_{K^\varepsilon}(t), B_{H^\varepsilon}(t), B^0_{\phi^\varepsilon}(t), B^0_{\psi^\varepsilon}(t), B^1_{\phi^\varepsilon}(t), B^1_{\psi^\varepsilon}(t), B^\infty_{\phi^\varepsilon}(t), B^\infty_{\psi^\varepsilon}(t)\\
&\le A\left(\left\|u^\varepsilon(t)\right\|_{m_6-1}\right) \le A\left(N^\varepsilon_{m_6}(t)\right).
\end{aligned}$$

We define

$$T_\varepsilon^* = \left\{0\le T < \frac{1/2\sqrt{ae}-\|u_0\|}{8C_2(N_{m_6})^3} \;\middle|\; N^\varepsilon_{m_6}(t) < 2N_{m_6}, 0\le t<T\right\},$$

where C_2 is a positive constant appearing in the estimate $\left\|f(u, \partial_x u, \partial_y u)\right\| \le C_2\|u\|_2^3$. Lemma 6.4.1 shows $T_\varepsilon^* > 0$. It is easy to see

$$\begin{aligned}\left\|u^\varepsilon(t)\right\| &\le \|u_0\| + \int_0^t \left\|f(u^\varepsilon(\tau), \partial_x u^\varepsilon(\tau), \partial_y u^\varepsilon(\tau))\right\| d\tau \\ &\le \|u_0\| + C_2\int_0^t \left\|u^\varepsilon(\tau)\right\|_2^3 d\tau \\ &\le \|u_0\| + 8C_2(N_{m_6})^3 t < \frac{1}{2\sqrt{ae}} \quad \text{for} \quad 0 \le t < T_\varepsilon^*.\end{aligned}$$

We here note that the local well-posedness ensures the validity of the energy estimates. Let n be an integer which ranges in $m_6 \le n \le m$. Using (6.25), we get

$$\begin{aligned}&\frac{d}{dt}\sum_{\substack{\alpha+\beta\le n\\ \alpha,\beta\le n-1}}\left\|\partial_x^\alpha\partial_y^\beta u^\varepsilon(t)\right\| \\ \le\ & 2\sum_{\substack{\alpha+\beta\le n\\ \alpha,\beta\le n-1}}\left\|\partial_x^\alpha\partial_y^\beta\left\{f(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\}\right\|\left\|\partial_x^\alpha\partial_y^\beta u^\varepsilon(t)\right\| \\ \le\ & C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2. \end{aligned} \tag{6.49}$$

(6.14) implies

$$\begin{aligned}&\frac{d}{dt}\left\|K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\right\| \\ \le\ & -4\Big(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\Big) \\ +\ & C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\ +\ & 2\mathrm{Re}\left(K^\varepsilon(t)\partial_x^n\left\{f(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\}, K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\right). \end{aligned} \tag{6.50}$$

We here decompose the last term of the right hand side of (6.50) as

$$\begin{aligned}\partial_x^n\left\{f(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\} &= g_5^n(u^\varepsilon(t)) + g_6^n(u^\varepsilon(t)), \\ g_5^n(u^\varepsilon(t)) &= a_0\partial_x^n\left\{f_0(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\} \\ &+ a_2\partial_x^n\left\{f_2(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\} \\ &+ a_1\left\{\partial_x^n\left\{f_1(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\}\right. \\ &- \left.\left(\int_y^{+\infty}\left(\partial_x^{n+1}u^\varepsilon(t)\bar{u}^\varepsilon(t) - u^\varepsilon(t)\partial_x^{n+1}\bar{u}^\varepsilon(t)\right)dy'\right)u^\varepsilon(t)\right\}, \\ g_6^n(u^\varepsilon(t)) &= a_1\left\{\left(\int_y^{+\infty}\left(\partial_x^{n+1}u^\varepsilon(t)\bar{u}^\varepsilon(t) + u^\varepsilon(t)\partial_x^{n+1}\bar{u}^\varepsilon(t)\right)dy'\right)u^\varepsilon(t)\right\}.\end{aligned}$$

Using (6.27) and (6.28), we have

$$2\mathrm{Re}\left(K^\varepsilon(t)\partial_x^n\left\{f(u^\varepsilon(t), \partial_x u^\varepsilon(t), \partial_y u^\varepsilon(t))\right\}, K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\right)$$

$$\begin{aligned}&\le C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\ &+ \frac{a}{M}\left(3 + e^{4M\|u^\varepsilon(t)\|^2}\right) \\ &\times \Big(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\Big) \\ &\le C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\ &+ \left(\frac{3}{e}+1\right) \\ &\times \Big(\phi^\varepsilon(t,x)\langle D_x 1/2K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x 1/2K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\Big)\end{aligned} \tag{6.51}$$

for $t \in [0, T_\varepsilon^*)$. (6.50) and (6.51) give

$$\begin{aligned}\frac{d}{dt}\left\|K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\right\| &\le C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2 \\ &- 3(1-1/e) \\ &\times \Big(\phi^\varepsilon(t,x)\langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t), \langle D_x\rangle^{1/2}K^\varepsilon(t)\partial_x^n u^\varepsilon(t)\Big) \\ &\le C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2.\end{aligned} \tag{6.52}$$

Similarly, using (6.15), (6.27) and (6.29), we get

$$\frac{d}{dt}\left\|H^\varepsilon(t)\partial_y^n u^\varepsilon(t)\right\| \le C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t)^2. \tag{6.53}$$

Combining (6.49), (6.52) and (6.53), we obtain

$$\frac{d}{dt}N_n^\varepsilon(t) \le C\|u^\varepsilon(t)\|_{n-1}^2 N_n^\varepsilon(t) \quad \text{for} \quad t \in [0, T_\varepsilon^*).$$

The Gronwall inequality implies

$$N_n^\varepsilon(t) \le N_n \exp\left(C\int_0^t \|u^\varepsilon(\tau)\|_{n-1}^2 d\tau\right) \quad \text{for} \quad t \in [0, T_\varepsilon^*). \tag{6.54}$$

When $n = m_6$ and $t = T_\varepsilon^*$, (6.54) becomes $2 \le \exp\left(4CT_\varepsilon^*(N_{m_6})^2\right)$. Noting the definition of T_ε^*, we obtain

$$T_\varepsilon^* \ge T \equiv \min\left(\frac{\log 2}{4C(N_{m_6})^2}, \frac{1/2\sqrt{ae} - \|u_0\|}{8C_2(N_{m_6})^3}\right) > 0.$$

This means that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^{m_6})$. Using (6.54) successively, we show that $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m)$. Then the standard compactness arguments imply that there exist a subsequence $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in L^\infty(0,T;H^m)$ such tnat

$$\begin{aligned}u^\varepsilon &\xrightarrow{\mathrm{W}^*} u \quad \text{in} \quad L^\infty(0,T;H^m) &&\text{as} \quad \varepsilon \downarrow 0, \\ u^\varepsilon &\longrightarrow u \quad \text{in} \quad C\left([0,T];H_{\mathrm{loc}}^{m-\delta}\right),\ \delta > 0 &&\text{as} \quad \varepsilon \downarrow 0.\end{aligned}$$

It is easy to see $u \in C_{\mathrm{W}}\big([0,T]; H^m\big)$. To see u is a solution to (1.1)–(1.2), we have only to check

$$f_j(u^\varepsilon, \partial_x u^\varepsilon, \partial_y u^\varepsilon) \longrightarrow f_j(u, \partial_x u, \partial_y u) \quad \text{in} \quad \mathcal{D}'\big((0,T)\times\mathbb{R}^2\big) \quad \text{as} \quad \varepsilon \downarrow 0, \quad j = 0,1,2. \tag{6.55}$$

It is easy to see the case of $j = 0$ in (6.55). We show the cases of $j = 1$. For this purpose, we introduce some classes of finite Radon measures as follows.

$$\mathcal{B}^0_0 \equiv C_0\big((0,T)\times\mathbb{R}^2\big) \quad \text{equipped with } \mathcal{B}^0 \text{ norm},$$

$$\begin{aligned}\tilde{\mathcal{B}}^0 \equiv \quad & \Big\{v(t,x,y) \in \mathcal{B}^0\big((0,T)\times\mathbb{R}^2\big) \;\Big|\; \exists R > 0, \exists w(t,x) \in C_0\big((0,T)\times(-R,+R)\big) \;\text{ s.t.}\\ & \operatorname{supp}[v] \subset (0,T)\times(-R,+R)\times(-R,+\infty), \quad v(t,x,y) = w(t,x) \;\text{ for }\; y > R\Big\}\\ & \quad \text{equipped with } \mathcal{B}^0 \text{ norm},\end{aligned}$$

$\mathcal{M} = (\mathcal{B}^0_0)'$ and $\tilde{\mathcal{M}} = (\tilde{\mathcal{B}}^0)'$. Clearly $\mathcal{B}^0_0$ and $\tilde{\mathcal{B}}^0$ are separable and not complete. $\mathcal{B}^0_0 \subset \tilde{\mathcal{B}}^0$ implies $\tilde{\mathcal{M}} \subset \mathcal{M}$. The properties of $\tilde{\mathcal{M}}$ are the following.

Lemma 6.4.2 *We assume that μ and ν belong to $\tilde{\mathcal{M}}$ and that $\mu = \nu$ in $\mathcal{M}'\big((0,T)\times\mathbb{R}^2\big)$. Then $\mu = \nu$ in $\tilde{\mathcal{M}}$.*

Proof. Lemma 6.4.2 is well known if we replace $\tilde{\mathcal{M}}$ by $L^1_{loc}\big((0,T)\times\mathbb{R}^2\big)$. Since $\mathcal{D}$ is dense in $\mathcal{B}^0_0$, $\mu = \nu$ in $\mathcal{D}'$ implies $\mu = \nu$ in $\mathcal{M}$. It is enough to consider the case of $\mu, \nu \geq 0$ and to prove

$$\langle \mu - \nu, v\rangle = 0 \quad \text{for any} \quad v \in \tilde{\mathcal{B}}^0, v \geq 0. \tag{6.56}$$

Let $\{\gamma_n\}_{n\geq 1} \subset C_0(\mathbb{R})$ satisfies $\gamma_n \geq 0$ and

$$\gamma_n(y) = \begin{cases} 1 & (|y| \leq n),\\ 0 & (|y| \geq n+1).\end{cases}$$

It is easy to see $\gamma_n(y)v(t,x,y) \in \mathcal{B}^0_0$ and

$$0 \leq \gamma_1(y)v(t,x,y) \leq \gamma_2(y)v(t,x,y) \leq \cdots \leq \gamma_n(y)v(t,x,y) \leq \cdots \longrightarrow v(t,x,y)$$

for any $v \in \mathcal{B}^0_0$ satisfying $v \geq 0$. We here note that we can see μ and ν as positive finite measures on Borel sets of $(0,T)\times\mathbb{R}^2$. Then $\mu = \nu$ in $\mathcal{M}$ means

$$\int_{(0,T)\times\mathbb{R}^2} \gamma_n(y)v(t,x,y)\,d\mu = \int_{(0,T)\times\mathbb{R}^2} \gamma_n(y)v(t,x,y)\,d\nu$$

for all $n \in \mathbb{N}$. Using the Beppo–Levi theorem, we obtain (6.56). ∎

Now we return to the proof of (6.55). Clearly

$$|u^\varepsilon|^2 \longrightarrow |u|^2 \quad \text{in} \quad \mathcal{D}'\big((0,T)\times\mathbb{R}^2\big) \quad \text{as} \quad \varepsilon \downarrow 0$$

and $|u|^2 \in \tilde{\mathcal{M}}$. Since $\big\{|u^\varepsilon|^2\big\}_{\varepsilon\in(0,1]}$ is bounded in $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{B}}^0$ is a separable normed vector space, there exist a subsequence $\big\{|u^\varepsilon|^2\big\}_{\varepsilon\in(0,1]}$ and $\mu \in \tilde{\mathcal{M}}$ such that

$$|u^\varepsilon| \xrightarrow{\mathrm{W}^*} \mu \quad \text{in} \quad \tilde{\mathcal{M}} \quad \text{as} \quad \varepsilon \downarrow 0.$$

$\tilde{\mathcal{M}} \subset \mathcal{D}'$ yields $\mu = |u|^2$ in $\mathcal{D}'$. Then Lemma 6.4.2 implies

$$|u^\varepsilon| \xrightarrow{\mathrm{W}^*} |u|^2 \quad \text{in} \quad \tilde{\mathcal{M}} \quad \text{as} \quad \varepsilon \downarrow 0. \tag{6.57}$$

Let $\alpha(t,x,y)$ belong to $\mathcal{D}\big((0,T)\times\mathbb{R}^2\big)$. Using the Fubini theorem and the integration by parts with respect to x, We have

$$\Big\langle f_1(u^\varepsilon, \partial_x u^\varepsilon, \partial_y u^\varepsilon) - f_1(u, \partial_x u, \partial_y u), \alpha\Big\rangle = F_1^\varepsilon(\alpha) + F_2^\varepsilon(\alpha),$$

$$\begin{aligned} F_1^\varepsilon(\alpha) &= \int_0^T \iint_{\mathbb{R}^2} \int_y^{+\infty} \partial_x \big|u^\varepsilon(t,x,y')\big|^2 dy'\\ &\qquad \times\big(u^\varepsilon(t,x,y) - u(t,x,y)\big)\alpha(t,x,y)\,dxdydt,\\ F_2^\varepsilon(\alpha) &= \int_0^T \iint_{\mathbb{R}^2} \Big(\big|u^\varepsilon(t,x,y')\big|^2 - \big|u(t,x,y')\big|^2\Big) v(t,x,y';\alpha)\,dxdy'dt,\\ v(t,x,y';\alpha) &= \partial_x \int_{-\infty}^{y'} u(t,x,y)\alpha(t,x,y)\,dy \in \tilde{\mathcal{B}}^0. \end{aligned}$$

It is easy to see $F_1^\varepsilon(\alpha) \to 0$ as $\varepsilon \downarrow 0$. $F_2^\varepsilon(\alpha) \to 0$ as $\varepsilon \downarrow 0$ follows from (6.57) directly. Then we have finished to prove the case of $j = 1$ in (6.55). Similarly we can show the case of $j = 2$ in (6.55). This completes the proof of the existence of a solution to (6.1)–(6.2).

The uniqueness of the solution can be proved by the same energy method as above. More precisely let $u, v \in L^\infty(0,T;H^m)$ be solutions to (6.1)–(6.2). We here note

$$\begin{aligned} & \partial_x u\bar{u} + u\partial_x\bar{u} - \partial_x v\bar{v} - v\partial_x\bar{v}\\ = \; & \partial_x(u-v)\bar{u} + \partial_x v(\overline{u-v}) + u\partial_x(\overline{u-v}) + (u-v)\partial_x\bar{v}. \end{aligned}$$

We define the symbol of the transformations $\partial_x(u-v) \mapsto K(t)\partial_x(u-v)$ and $\partial_y(u-v) \mapsto H(t)\partial_y(u-v)$ by using u and we evaluate $\|u-v\|_1$. These procedure imply the uniqueness of the solution. This completes the proof of Theorem 6.1.1.

6.5 Proof of global existence theorem

Finally we prove Theorem 6.1.2 by the *a priori* estimates. We take $m_7 \in \mathbb{N}$ as $m_7 = \max(l+4, 6) \geq m_1 + 1$. In view of Theorem 6.1.1, we have only to obtain the *a priori* estimate of $\|u(t)\|_{m_7-1}$ $(< 1/2\sqrt{ae})$. In the same way as the previous section, we define $K(t)$, $H(t)$, $B_K(t)$ and etc. We denote $Y(t)$ by

$$\begin{aligned} Y(t) &= \sum_{j=0}^{4} X_{m_7-1-j,j}(t) + (1+t)^{-1/2} X_{m_7-6,5}(t) \\ &+ \sum_{j=0}^{4} \sum_{\substack{\alpha+\beta=m_7-j \\ \alpha'+\beta'=j \\ \alpha+\alpha',\beta+\beta' \leq m_7-1}} \left\| \partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} u(t) \right\| \\ &+ (1+t)^{-1/2} \Big(\sum_{\substack{\alpha+\alpha'=m_7 \\ \alpha' \leq 5}} \left\| K(t) \partial_x^\alpha J_x^{\alpha'} u(t) \right\| + \sum_{\substack{\beta+\beta'=m_7 \\ \beta' \leq 5}} \left\| H(t) \partial_x^\beta J_x^{\beta'} u(t) \right\| \Big). \end{aligned}$$

We suppose

$$\sup_{t \in [0,T)} Y(t) \leq R \quad \text{for some} \quad T > 0,$$

where $R > 0$ is smaller than $1/2\sqrt{ae}$. We have

$$\begin{aligned} &\left| \int_0^x \partial_t \phi(t, x') dx' \right| \\ &= M \left| \int_0^x \int_{-\infty}^{+\infty} \partial_t \left| u(t, x', y) \right|^2 dy dx' \right| \\ &= M \left| \int_0^x \int_{-\infty}^{+\infty} (\partial_t u \bar{u} + u \partial_t \bar{u}) dy dx' \right| \\ &= M \left| \int_0^x \int_{-\infty}^{+\infty} \Big\{ i(\partial_x^2 + \partial_y^2) u \bar{u} - iu (\partial_x^2 + \partial_y^2) \bar{u} \right. \\ &\qquad \left. + f(u, \partial_x u, \partial_y u) \bar{u} + \overline{f(u, \partial_x u, \partial_y u)} u \Big\} dy dx' \right| \\ &\leq M \left| \int_0^x \int_{-\infty}^{+\infty} (\partial_x u \bar{u} + u \partial_x \bar{u}) dy dx' \right| + M \left\| f(u(t)) \right\|_{L^\infty} \|u(t)\|^2 \\ &\leq C(1+t)^{-1} \sum_{j=0}^{2} X_{2-j,j}(t)^2 \left(1 + \|u(t)\|^2\right) \\ &\leq C(1+t)^{-1} Y(t)^2. \end{aligned}$$

This means that the transformation $u(t) \mapsto K(t)u(t)$ does not bring about the loss of time-decay because of the structural nice property of $\phi(t,x)$. This was first pointed out by S. Katayama and Y. Tsutsumi ([27]). Similarly we have

$$\begin{aligned} B_K(t), B_H(t), B_\phi^0(t), B_\psi^0(t) &\leq C, \\ B_\phi^1(t), B_\psi^1(t), B_\phi^\infty(t), B_\psi^\infty(t) &\leq CR^2(1+t)^{-1}. \end{aligned}$$

(6.36), (6.37), (6.38) and (6.40) become

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta' \leq m_7 \\ \alpha',\beta' \leq 5 \\ \alpha+\alpha',\beta+\beta' \leq m_7-1 \\ \alpha',\beta' \leq 4}} \left\| \partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} \big\{ f(u(t), \partial_x u(t), \partial_y u(t)) \big\} \right\| \leq CR^3 (1+t)^{-3/2}, \tag{6.58}$$

$$\begin{aligned} &\sum_{\alpha+\beta \leq m_7-6} \Big\{ \left\| \partial_x^\alpha \partial_y^\beta J_x^5 \big\{ f(u(t), \partial_x u(t), \partial_y u(t)) \big\} \right\| \\ &\qquad + \left\| \partial_x^\alpha \partial_y^\beta J_y^5 \big\{ f(u(t), \partial_x u(t), \partial_y u(t)) \big\} \right\| \Big\} \\ &\leq CR^3 (1+t)^{-1/2}, \end{aligned} \tag{6.59}$$

$$\begin{aligned} &\sum_{\substack{\alpha+\alpha'=m_7 \\ \alpha' \leq 5}} \left\| \Big(K(t) \partial_x^\alpha J_x^{\alpha'} \big\{ f(u(t), \partial_x u(t), \partial_y u(t)) \big\}, K(t) \partial_x^\alpha J_x^{\alpha'} u(t) \Big) \right\| \\ &\leq CR^4 + \frac{1}{2}\left(1 + \frac{3}{e}\right) \\ &\times \sum_{\substack{\alpha+\alpha'=m_7 \\ \alpha' \leq 5}} \Big(\phi(t,x) \langle D_x \rangle^{1/2} K(t) \partial_x^\alpha J_x^{\alpha'} u(t), \langle D_x \rangle^{1/2} K(t) \partial_x^\alpha J_x^{\alpha'} u(t) \Big), \end{aligned} \tag{6.60}$$

$$\begin{aligned} &\sum_{\substack{\beta+\beta'=m_7 \\ \beta' \leq 5}} \left\| \Big(H(t) \partial_y^\beta J_y^{\beta'} \big\{ f(u(t), \partial_x u(t), \partial_y u(t)) \big\}, H(t) \partial_y^\beta J_y^{\beta'} u(t) \Big) \right\| \\ &\leq CR^4 + \frac{1}{2}\left(1 + \frac{3}{e}\right) \\ &\times \sum_{\substack{\beta+\beta'=m_7 \\ \beta' \leq 5}} \Big(\psi(t,y) \langle D_y \rangle^{1/2} H(t) \partial_y^\beta J_y^{\beta'} u(t), \langle D_y \rangle^{1/2} H(t) \partial_y^\beta J_y^{\beta'} u(t) \Big). \end{aligned} \tag{6.61}$$

Using (6.14), (6.15), (6.60) and (6.61), we get

$$\frac{d}{dt} \Big(\sum_{\substack{\alpha+\alpha'=m_7 \\ \alpha' \leq 5}} \left\| K(t) \partial_x^\alpha J_x^{\alpha'} u(t) \right\|^2 + \sum_{\substack{\beta+\beta'=m_7 \\ \beta' \leq 5}} \left\| H(t) \partial_y^\beta J_y^{\beta'} u(t) \right\|^2 \Big) \leq CR^4.$$

Then we have

$$(1+t)^{-1/2} \Big(\sum_{\substack{\alpha+\alpha'=m_7 \\ \alpha' \leq 5}} \left\| K(t) \partial_x^\alpha J_x^{\alpha'} u(t) \right\| + \sum_{\substack{\beta+\beta'=m_7 \\ \beta' \leq 5}} \left\| H(t) \partial_y^\beta J_y^{\beta'} u(t) \right\| \Big) \leq C(\delta + R^2). \tag{6.62}$$

On the other hand, (6.58) and (6.59) yield

$$\sum_{\substack{\alpha+\beta+\alpha'+\beta' \leq m_7 \\ \alpha',\beta' \leq 5 \\ \alpha+\alpha',\beta+\beta' \leq m_7-1 \\ \alpha',\beta' \leq 4}} \left\| \partial_x^\alpha \partial_y^\beta J_x^{\alpha'} J_y^{\beta'} u(t) \right\| \leq C(\delta + R^3), \tag{6.63}$$

$$(1+t)^{-1/2} \sum_{\alpha+\beta \le m_7-6} \left\{ \left\| \partial_x^\alpha \partial_y^\beta J_x^5 u(t) \right\| + \left\| \partial_x^\alpha \partial_y^\beta J_y^5 u(t) \right\| \right\} \le C(\delta + R^3). \tag{6.64}$$

Combining (6.62), (6.63) and (6.64), we obtain

$$\sup_{t \in [0,T)} Y(t) \le C(\delta + R^2).$$

Let R_1 be a positive small constant satisfying $CR_1^2 \le R_1/4$. We can take R as $R \le R_1$. Then we have

$$\sup_{t \in [0,T)} Y(t) \le \frac{R}{2}$$

provided that δ is sufficiently small. This completes the proof of Theorem 6.1.2.

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